On the Norm of a Composition Operator

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Abstract

Let φ be an analytic map that takes the open unit disk of $\mathbb{C}$ into itself. The composition operator C_φ, acting on either the Hardy space H^2 or one of the weighted Bergman spaces A^2_α, is defined by the rule

$$C_\varphi(f) = f \circ \varphi.$$

Every composition operator is bounded on each of these spaces, but the exact value of its norm is rarely known. In view of this situation, we consider several problems relating to this basic, yet elusive question. Our strategy for studying $\|C_\varphi\|$ centers around determining the spectrum of the operator $C_\varphi^* C_\varphi$. We obtain our most substantial results in the context of H^2, in particular for a composition operator induced by a linear fractional map:

- We determine the conditions under which $\|C_\varphi\|$ is given by the action of either C_φ or C_φ^* on the reproducing kernel functions of H^2.
- Assuming a specific set of conditions on φ, we describe $\|C_\varphi\|$ in terms of the zeros of a particular polynomial.
- We obtain several results, in the case where $\varphi(0) = 0$, pertaining to the norm of C_φ acting on various subspaces of H^2.

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Contents

Introductory Remarks

This thesis considers questions relating to the norm of a composition operator C_φ acting on various Hilbert spaces of analytic functions.

- Chapter 1 defines our basic objects of study: functional Hilbert spaces, reproducing kernel functions, and composition operators. We outline our general strategy for determining the norm of an operator. The quantities S_φ and S_φ^* are discussed for the first time.
- Chapter 2 introduces the particular Hilbert spaces in which we are most interested: the Hardy space H^2 and the weighted Bergman spaces A^2_α. We discuss the essential norm of a composition operator acting on one of these spaces, as well as the quantities S_φ and S_φ^*. We briefly consider multiplication operators and define the Banach spaces H^p and A^p_α.
- Chapter 3, which deals with composition operators induced by linear fractional maps, provides the foundation for much of our subsequent work. Having derived Cowen's adjoint formula, we establish a useful representation for the operator $C_\varphi^* C_\varphi : H^2 \to H^2$. We apply this result to obtain information about S_φ and S_φ^*.

- Chapter 4 considers the spectrum of an operator which "locally resembles" a weighted composition operator. We use these results to determine the norm of a composition operator whose symbol has the form $\varphi(z) = az + b$.

- Chapter 5 contains the most significant results of this thesis. We introduce a class of linear fractional maps and determine the norm of a corresponding operator $C_\varphi : H^2 \to H^2$. We identify the elements of H^2 on which such an operator attains its norm and give specific examples of maps belonging to this class. Several related results are also discussed.

- Chapter 6 considers the norm of a composition operator acting on various subspaces of H^2. We pay particular attention to the situation where φ is a linear fractional map.

Chapter 1

Composition Operators on Functional Hilbert Spaces

This chapter introduces the general concepts considered in this thesis: functional Hilbert spaces, reproducing kernel functions, and composition operators. Section 1.2 outlines our basic strategy for obtaining results about the norm of an operator. The chapter concludes with a brief discussion of S_φ and S_φ^*, the quantities we use to describe the action of a composition operator (and its adjoint) on the reproducing kernel functions of a functional Hilbert space.

1.1 Definitions

Let $\mathcal{H}$ be a nontrivial Hilbert space whose elements are functions on a set U. We call $\mathcal{H}$ a *functional Hilbert space* if every point-evaluation functional is bounded on $\mathcal{H}$; that is, for every point w in U, the map $\lambda_w : \mathcal{H} \to \mathbb{C}$ defined by the rule

$$\lambda_w(f) = f(w)$$

is continuous. In this case, for each w in U, the Riesz representation theorem guarantees the existence of a unique element K_w of $\mathcal{H}$ such that

$$\langle f, K_w \rangle = f(w)$$

for every f in $\mathcal{H}$. These elements K_w are called the *reproducing kernel functions* for $\mathcal{H}$. Provided that we have an explicit representation for K_w, it is always easy to compute its norm in $\mathcal{H}$:

$$\|K_w\| = \sqrt{\langle K_w, K_w \rangle} = \sqrt{K_w(w)}.$$

We write k_w to denote the normalized kernel function $K_w/\|K_w\|$.

Any element f of $\mathcal{H}$ that is orthogonal to each K_w has the property that

$$f(w) = \langle f, K_w \rangle = 0$$

for every w in U, which means that f is identically 0; in other words, the kernel functions span a dense subset of $\mathcal{H}$. In general, for a subset W of U, we write $\mathcal{K}_W$ to denote the closed linear span of the kernel functions $\{K_w : w \in W\}$. The orthogonal complement $\mathcal{K}_W^{\perp}$ is precisely the set of all functions in $\mathcal{H}$ that vanish on W.

The term *reproducing kernel Hilbert space* is a common synonym for *functional Hilbert space.* If the elements of a functional Hilbert space $\mathcal{H}$ happen to be analytic functions on an open subset U of $\mathbb{C}^n$, we refer to $\mathcal{H}$ as a *Hilbert space of analytic functions.*

Consider a functional Hilbert space $\mathcal{H}$ on a set U. Let φ be a *self-map* of U, that is, a map taking U into U. The *composition operator* C_φ on $\mathcal{H}$ is defined by the rule

$$C_\varphi(f) = f \circ \varphi.$$

The map φ is referred to as the *symbol* of the composition operator C_φ. It is easy to see that every composition operator is linear; moreover, we make the following observation:

Proposition 1.1 *Any composition operator that takes $\mathcal{H}$ into $\mathcal{H}$ is automatically bounded.*

Proof. We shall appeal to the closed graph theorem. Suppose that $\{f_n\}$ is a sequence of functions in $\mathcal{H}$ converging in norm to an element f; suppose also that $\{C_\varphi(f_n)\}$ converges to some element g. Since each point-evaluation functional is bounded, the functions f_n converge pointwise to f; therefore the functions $C_\varphi(f_n) = f_n \circ \varphi$ converge pointwise to $C_\varphi(f) = f \circ \varphi$. The sequence $\{C_\varphi(f_n)\}$ must also converge pointwise to g, which implies that $g = C_\varphi(f)$; hence our assertion follows. $\square$

While it is easy to describe a composition operator C_φ, it is often difficult to represent the adjoint C_φ^* explicitly. The adjoint does, however, have one particularly useful property. Suppose that C_φ is bounded on a functional Hilbert space $\mathcal{H}$; observe that

$$\left\langle f, C_\varphi^*(K_w) \right\rangle = \langle C_\varphi(f), K_w \rangle = \langle f \circ \varphi, K_w \rangle = f(\varphi(w)) = \langle f, K_{\varphi(w)} \rangle$$

for each f in $\mathcal{H}$. Consequently $C_\varphi^*(K_w) = K_{\varphi(w)}$ for every kernel function K_w.

1.2 The norm of an operator

Suppose that T is a bounded operator on a Hilbert (or Banach) space $\mathcal{H}$. As long as there is no ambiguity regarding the space on which T is acting, we simply write $\|T\|$ to denote the norm of T; we write $\|T : \mathcal{H} \to \mathcal{H}\|$ whenever we want to specify the underlying space explicitly. The main results of this thesis pertain to norms of composition operators acting on various Hilbert spaces of analytic functions. In this section, we develop a general method for dealing with questions relating to norms.

Let T be a bounded operator on a Hilbert space $\mathcal{H}$. One reasonable strategy for determining $\|T\|$ is to investigate the spectrum of the operator T^*T. Since T^*T is self-adjoint, its spectral radius equals $\|T^*T\| = \|T\|^2$. The following observation underscores the connection between the spectrum of T^*T and the norm of T:

Proposition 1.2 *Let h be an element of $\mathcal{H}$; then $\|T(h)\| = \|T\|\,\|h\|$ if and only if $(T^*T)(h) = \|T\|^2 h$.*

This proposition can be proved with a straightforward Hilbert space argument, as shown below, or can be deduced from other well-known results (e.g. Theorem III.43 in [17]).

Proof of Proposition 1.2. Suppose that $(T^*T)(h) = \|T\|^2 h$; then

$$\|T(h)\|^2 = \langle T(h), T(h)\rangle = \langle (T^*T)(h), h\rangle = \left\langle \|T\|^2 h, h\right\rangle = \|T\|^2 \|h\|^2 ,$$

as we had hoped to show.

Conversely, suppose that $\|T(h)\| = \|T\|\,\|h\|$; then

$$\begin{aligned} \|T\|^2 \|h\|^2 &= \|T(h)\|^2 = \langle T(h), T(h)\rangle = \langle (T^*T)(h), h\rangle \\ &\leq \|(T^*T)(h)\|\,\|h\| \leq \|T\|^2 \|h\|^2 . \end{aligned}$$

In particular, the first inequality is an equality, which means that $(T^*T)(h)$ is a scalar multiple of h; since $\|(T^*T)(h)\| = \|T\|^2 \|h\|$ and T^*T is a positive operator, we conclude that $(T^*T)(h) = \|T\|^2 h$. □

Whenever $\|T(h)\| = \|T\|\,\|h\|$ for $h \neq 0$, we say that the operator T *attains its norm* on the element h.

Let $\mathcal{B}(\mathcal{H})$ denote the Banach algebra consisting of all bounded operators on $\mathcal{H}$; let $\mathcal{B}_0(\mathcal{H})$ denote the closed ideal consisting of the compact operators. The quotient $\mathcal{B}(\mathcal{H})/\mathcal{B}_0(\mathcal{H})$ is often called the *Calkin algebra*. We write $\|T\|_e$ (or $\|T : \mathcal{H} \to \mathcal{H}\|_e$) to denote the *essential norm* of an operator T in $\mathcal{B}(\mathcal{H})$; that is,

$$\|T\|_e = \inf_{K \in \mathcal{B}_0(\mathcal{H})} \|T - K\|.$$

The essential norm represents the distance between T and the compact operators; in particular, $\|T\|_e = 0$ if and only if T itself is compact. One can also view $\|T\|_e$ as being the norm of the coset $T + \mathcal{B}_0(\mathcal{H})$ in the Calkin algebra. The *essential spectrum* of T is simply defined to be the spectrum of the element $T + \mathcal{B}_0(\mathcal{H})$ in $\mathcal{B}(\mathcal{H})/\mathcal{B}_0(\mathcal{H})$; the *essential spectral radius* of T, which we write $r_e(T)$, is the largest modulus belonging to a point in the essential spectrum of T. One can easily show that the essential spectrum of T is contained in the spectrum of T. Whenever T is a normal operator, any point in the spectrum of T that does not belong to the essential spectrum must be an eigenvalue of finite multiplicity (see Proposition XI.4.6 in [8]).

It is not difficult to show that $\|T^*T\|_e = \|T\|_e^2$ for any bounded operator T; from this fact, we can deduce that $r_e(T) = \|T\|_e$ whenever T is self-adjoint. Similarly, recall that $r(T)$, the spectral radius of T, equals $\|T\|$ for any self-adjoint operator. In light of Proposition 1.2, our next observation follows easily:

Proposition 1.3 *If $\|T\|_e < \|T\|$, then T attains its norm on an element of $\mathcal{H}$.*

Proof. Consider the positive operator T^*T; observe that

$$r_e(T^*T) = \|T^*T\|_e = \|T\|_e^2 < \|T\|^2 = \|T^*T\| = r(T^*T).$$

Therefore $\|T\|^2$, the largest element of the spectrum of T^*T, does not belong to the essential spectrum; since any self-adjoint operator is normal, $\|T\|^2$ must be an eigenvalue of finite multiplicity. Consequently T^*T has an eigenvector corresponding to $\|T\|^2$, on which the operator T attains its norm. $\square$

It is helpful to remember Proposition 1.3 when studying composition operators, particularly since it is often easier to compute $\|C_\varphi\|_e$ than $\|C_\varphi\|$. As it happens, in most of the cases where we already know $\|C_\varphi\|$ exactly (see Sections 2.1 and 2.2), the operator has the property that $\|C_\varphi\|_e = \|C_\varphi\|$, a condition sometimes called *extremal noncompactness*.

1.3 The quantities S_φ and S_φ^*

Let $\mathcal{H}$ be a functional Hilbert space on a set U; suppose that φ is a self-map of U that induces a bounded composition operator on $\mathcal{H}$. Recall that $C_\varphi^*(K_w) = K_{\varphi(w)}$ for any kernel function K_w; it seems reasonable, when investigating $\|C_\varphi\|$, to consider the action of C_φ^* on the reproducing kernel functions of $\mathcal{H}$.

Define N to be the set of all w in U such that $f(w) = 0$ for each f in $\mathcal{H}$. (For many spaces, including those introduced in Chapter 2, the set N is empty. We shall only encounter a few situations where N is nonempty: the spaces $H^2_{(n)}$, as defined in Chapter 6, have the property that $N = \{0\}$ whenever $n \geq 1$.) Since C_φ takes $\mathcal{H}$ into $\mathcal{H}$, the set $\varphi(N)$ must be contained in N. The kernel function K_w is identically zero if and only if the point w belongs to N. We define the quantity

$$\begin{aligned} S_\varphi^* &= \sup_{w\in U\setminus N} \left\{ \|C_\varphi^*(k_w)\| \right\} = \sup_{w\in U\setminus N} \left\{ \frac{\|C_\varphi^*(K_w)\|}{\|K_w\|} \right\} \\ &= \sup_{w\in U\setminus N} \left\{ \frac{\|K_{\varphi(w)}\|}{\|K_w\|} \right\} = \sup_{w\in U\setminus N} \left\{ \sqrt{\frac{K_{\varphi(w)}(\varphi(w))}{K_w(w)}} \right\}. \end{aligned}$$

Since $C_\varphi^*(K_w) = K_{\varphi(w)}$, the set $\{K_w : w \in \varphi^{-1}(N)\}$ consists precisely of those reproducing kernel functions which belong to the kernel of C_φ^*. Therefore

$$S_\varphi^* = \sup_{w\in U\setminus N} \left\{ \frac{\|C_\varphi^*(K_w)\|}{\|K_w\|} \right\} = \sup_{w\in U\setminus \varphi^{-1}(N)} \left\{ \frac{\|C_\varphi^*(K_w)\|}{\|K_w\|} \right\},$$

where the latter supremum is taken to be 0 if $U = \varphi^{-1}(N)$.

In similar fashion, we define

$$S_\varphi = \sup_{w\in U\setminus N} \{\|C_\varphi(k_w)\|\} = \sup_{w\in U\setminus N} \left\{ \frac{\|C_\varphi(K_w)\|}{\|K_w\|} \right\}.$$

The values of S_φ and S_φ^* serve to quantify the actions of C_φ and C_φ^* on the set of reproducing kernel functions. Whenever $S_\varphi = \|C_\varphi\|$, the norm of C_φ is given by the action of the operator on the kernel functions; likewise, when $S_\varphi^* = \|C_\varphi\|$, the norm is given by the action of C_φ^* on the kernel functions. While S_φ^* is generally easier to compute, the quantity S_φ is more likely to equal $\|C_\varphi\|$. In fact, we have the following result:

Proposition 1.4 *Let C_φ be a bounded composition operator on a functional Hilbert space $\mathcal{H}$; then $S_\varphi^* \le S_\varphi$.*

Proof. We begin with a general observation. Let T be a bounded operator on $\mathcal{H}$; then, for any element f not in the kernel of T,

$$\frac{\|T(f)\|}{\|f\|} = \frac{\|T(f)\|^2}{\|T(f)\|\,\|f\|} = \frac{\langle (T^*T)(f), f\rangle}{\|T(f)\|\,\|f\|} \le \frac{\|(T^*T)(f)\|\,\|f\|}{\|T(f)\|\,\|f\|} = \frac{\|T^*(T(f))\|}{\|T(f)\|}. \tag{1.1}$$

If $U = \varphi^{-1}(N)$, then $S_\varphi^* = 0$; thus the claim of the proposition is trivially true. Otherwise, applying (1.1) with $T = C_\varphi^*$ and f from the set $\{K_w : w \in U \setminus \varphi^{-1}(N)\}$, we see that

$$\begin{aligned} S_\varphi^* &= \sup_{w \in U \setminus \varphi^{-1}(N)} \left\{ \frac{\|C_\varphi^*(K_w)\|}{\|K_w\|} \right\} \leq \sup_{w \in U \setminus \varphi^{-1}(N)} \left\{ \frac{\|C_\varphi\left(C_\varphi^*(K_w)\right)\|}{\|C_\varphi^*(K_w)\|} \right\} \\ &= \sup_{w \in U \setminus \varphi^{-1}(N)} \left\{ \frac{\|C_\varphi\left(K_{\varphi(w)}\right)\|}{\|K_{\varphi(w)}\|} \right\} \leq S_\varphi, \end{aligned}$$

as we had hoped to show. □

The quantities S_φ and S_φ^*, under various names, have received a good deal of attention in recent years. Several authors ([1], [2], [6], and [23]) have studied these quantities in the context of the Hardy space H^2. Proposition 1.4 in this section is a straightforward generalization of Proposition 3.1 in [6]; the calculation of line (1.1) appears as Proposition 2.15 in [23]. We shall revisit the quantities S_φ and S_φ^* in Sections 2.4 and 3.3.

Chapter 2

Hardy Spaces and Weighted Bergman Spaces

Let $\mathbb{D}$ denote the open unit disk in the complex plane. For the remainder of this thesis, we work with various Hilbert spaces of analytic functions on $\mathbb{D}$. In particular, we consider composition operators acting on the Hardy space H^2 (the space for which we obtain the most interesting results) and the weighted Bergman spaces A^2_α. Sections 2.1 and 2.2 serve as a brief introduction to this material, providing relevant definitions and general results. Section 2.3 summarizes much of what is known about the essential norms of composition operators acting on these spaces. In Section 2.4, we address the quantities S_φ and S^*_φ in the particular context of H^2 and A^2_α; much of this section is a generalization of the work of Bourdon and Retsek [6]. The last two sections of this chapter deal with topics which, although somewhat detached from the principal concerns of this thesis, pertain to the overall issues in which we are interested.

2.1 The Hardy space H^2

The *Hardy space* H^2 is the Hilbert space consisting of all analytic functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$ on $\mathbb{D}$ with

$$\|f\|_2^2 = \sum_{n=0}^{\infty} |a_n|^2 < \infty,$$

where the inner product of two functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ is defined

$$\langle f, g\rangle = \sum_{n=0}^{\infty} a_n \overline{b_n}.$$

There are several other ways to express the Hardy space inner product: for example,

$$\langle f, g\rangle = \lim_{r\uparrow 1} \int_0^{2\pi} f\big(re^{i\theta}\big)\overline{g\big(re^{i\theta}\big)}\,\frac{d\theta}{2\pi}$$

or

$$\langle f, g\rangle = \int_0^{2\pi} f\big(e^{i\theta}\big)\overline{g\big(e^{i\theta}\big)}\,\frac{d\theta}{2\pi},$$

where f and g are defined almost everywhere on $\partial\mathbb{D}$ via their radial limits (see, for example, Sections 2.1 and 2.3 of [13]).

Every point-evaluation functional is bounded on H^2, which means that H^2 is a Hilbert space of analytic functions on $\mathbb{D}$. The reproducing kernel functions for the Hardy space are easy to describe:

$$K_w(z) = \sum_{n=0}^{\infty} \overline{w}^n z^n = \frac{1}{1-\overline{w}z}.$$

Therefore

$$\|K_w\|_2 = \sqrt{K_w(w)} = \sqrt{\frac{1}{1-|w|^2}}$$

and

$$k_w(z) = \frac{K_w(z)}{\|K_w\|_2} = \frac{\sqrt{1-|w|^2}}{1-\overline{w}z}.$$

The Hardy space has proved to be an especially popular environment for the study of composition operators. The first publications on the subject ([20] and [25]) dealt exclusively with composition operators on H^2 (or the related Banach spaces H^p). While mathematicians have subsequently considered composition operators acting on a variety of spaces, the Hardy space still remains a particular focus of attention. As we shall see, the techniques we use to prove our most interesting results depend on properties which are peculiar to H^2.

Any analytic map $\varphi : \mathbb{D} \to \mathbb{D}$ induces a bounded composition operator on H^2; in fact,

$$\sqrt{\frac{1}{1-|\varphi(0)|^2}} \leq \left\| C_\varphi : H^2 \to H^2 \right\| \leq \sqrt{\frac{1+|\varphi(0)|}{1-|\varphi(0)|}}. \tag{2.1}$$

(See, for example, Corollary 3.7 in [11].) The upper bound follows from the Littlewood subordination theorem [18]; the lower bound can be deduced from an argument similar to that of Lemma 2.12. In general, though, there is no known procedure for computing the norm of C_φ on H^2. We see from (2.1) that $\|C_\varphi\| = 1$ whenever $\varphi(0) = 0$. There are only a few other examples where we can determine the norm exactly; prior to the author's research, $\|C_\varphi\|$ was known for the following cases:

(1) φ is inner; that is, $\lim_{r\uparrow 1} \left|\varphi\bigl(re^{i\theta}\bigr)\right| = 1$ for almost all θ in $[0, 2\pi)$,

(2) $\varphi(z) = az + b$ where $|a| + |b| \leq 1$,

(3) $\varphi(z) = \frac{(r+s)z+(1-s)}{r(1-s)z+(1+sr)}$ where $0 < s < 1$ and $0 \leq r \leq 1$.

Case (1) was established by Nordgren [20]. Cowen [9] was the first to calculate the norm in case (2); we provide a new proof of his result in Section 4.2. We can compute the norms in case (3) because the adjoints of the corresponding composition operators are subnormal, a result due to Cowen and Kriete [10]. Cases (1) and (3) have the property that $\|C_\varphi\| = \|C_\varphi\|_e$, as does case (2) when $|a| + |b| = 1$.

The inequalities in (2.1) are sharp; for any value of $\varphi(0)$, there are particular examples of φ for which $\|C_\varphi\|$ equals the upper bound and examples for which $\|C_\varphi\|$ equals the lower bound. As it turns out, there are only a few maps φ for which $\|C_\varphi\|$ achieves either extreme. When $\varphi(0) = 0$, the norm of C_φ equals both the upper and lower bound; suppose then that $\varphi(0) \neq 0$. In this case, Joel Shapiro [28] showed that $\|C_\varphi\|$ equals the upper bound if and only if φ is an inner function. On the other hand, $\|C_\varphi\|$ equals the lower bound if and only if φ is a constant map, as we shall show in Lemma 2.12.

In the aforementioned cases where we know $\|C_\varphi\|$, the norm is given by the action of the operator on the set of reproducing kernel functions; that is, $\|C_\varphi\| = S_\varphi$. Moreover, $\|C_\varphi\| = S_\varphi^*$ in those cases where φ is univalent. This situation, however, does not hold in general, a fact first proved by Appel, Bourdon, and Thrall [1]. These authors considered the linear fractional map $\varphi(z) = 2/(3 - z)$; while they did not calculate $\|C_\varphi\|$ exactly, they did demonstrate that $S_\varphi^* < S_\varphi < \|C_\varphi\|$ for this particular φ. Bourdon and Retsek [6], in fact, showed that it is quite rare for S_φ^* to equal $\|C_\varphi\|$, a result we shall restate as Proposition 2.10.

Our main results for composition operators on H^2 pertain to the situation where

$$\varphi(z) = \frac{az + b}{cz + d}$$

is a nonconstant linear fractional self-map of $\mathbb{D}$. For such φ, we determine the conditions under which either $S_\varphi = \|C_\varphi\|$ or $S_\varphi^* = \|C_\varphi\|$ (Theorem 3.7). We introduce a new set of conditions on φ under which, at least in principle, we can calculate $\|C_\varphi\|$ (Theorem 5.11). Assuming these conditions, we identify the elements of H^2 on which C_φ attains its norm, each of which is a finite linear combination of kernel functions. We also obtain some interesting results, in the case where $\varphi(0) = 0$, pertaining to the norm of C_φ acting on various subspaces of H^2 (Theorems 6.10 and 6.12).

Before we turn our attention to other Hilbert spaces of analytic functions, we mention a few results which are specific to composition operators on H^2. The first, which we shall require on several occasions, is originally due to Nordgren [20]:

Proposition 2.1 *Let* $\varphi : \mathbb{D} \to \mathbb{D}$ *be an inner function with* $\varphi(0) = 0$. *The operator* $C_\varphi : H^2 \to H^2$ *is an isometry of* H^2.

Our proof is identical to that presented in [11].

Proof of Proposition 2.1. Since φ is an inner function, $\overline{\varphi(e^{i\theta})} = \left(\varphi(e^{i\theta})\right)^{-1}$ for almost all θ in $[0, 2\pi)$. Therefore $\|\varphi^n\|_2 = 1$ for all n and

$$\langle \varphi^n, \varphi^m \rangle = \int_0^{2\pi} \left(\varphi(e^{i\theta})\right)^n \overline{\left(\varphi(e^{i\theta})\right)^m} \, \frac{d\theta}{2\pi} = \int_0^{2\pi} \left(\varphi(e^{i\theta})\right)^{n-m} \frac{d\theta}{2\pi} = \left(\varphi(0)\right)^{n-m} = 0$$

whenever $n > m$. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be an arbitrary element of H^2; observe that

$$\|f \circ \varphi\|_2^2 = \left\langle \sum_{n=0}^{\infty} a_n \varphi^n, \sum_{m=0}^{\infty} a_m \varphi^m \right\rangle = \sum_{n=0}^{\infty} |a_n|^2 = \|f\|_2^2 .$$

In other words, the operator $C_\varphi : H^2 \to H^2$ is an isometry. ☐

The next result pertains to the elements of H^2 on which a composition operator attains its norm:

Proposition 2.2 *Suppose that the operator* $C_\varphi : H^2 \to H^2$ *attains its norm on an element* g *of* H^2. *If* φ *is not an inner function, then* g *cannot vanish at any point* w *of* $\mathbb{D}$.

Proof. Suppose that $g(w) = 0$ for some w in $\mathbb{D}$. Consider the Blaschke factor

$$b_w(z) = \frac{w - z}{1 - \overline{w} z}.$$

This function is an isometric zero-divisor for H^2; that is, the function

$$\widetilde{g}(z) = \frac{g(z)}{b_w(z)}$$

belongs to H^2, with $\|\widetilde{g}\|_2 = \|g\|_2$. Because φ is not an inner function, neither is the composition $b_w \circ \varphi$. Since $\lim_{r\uparrow 1} g\big(re^{i\theta}\big)$ is nonzero for almost all θ, we see that

$$\lim_{r\uparrow 1}\left|\frac{g\big(\varphi\big(re^{i\theta}\big)\big)}{b_w\big(\varphi\big(re^{i\theta}\big)\big)}\right| > \lim_{r\uparrow 1}\left|g\big(\varphi\big(re^{i\theta}\big)\big)\right|$$

for θ in a set of positive measure. Hence $\|C_\varphi(\widetilde{g})\|_2 > \|C_\varphi(g)\|_2$, contradicting our choice of g. □

Corollary 2.3 *Suppose that φ is not inner; if g_1 and g_2 are functions on which C_φ attains its norm, then one is a scalar multiple of the other.*

Proof. Both g_1 and g_2 are eigenfunctions for $C_\varphi^* C_\varphi : H^2 \to H^2$ corresponding to the eigenvalue $\|C_\varphi\|^2$; moreover, $g_1(0)$ and $g_2(0)$ are both nonzero. If $g_1 - (g_1(0)/g_2(0))\, g_2$ were not identically 0, then it would be an eigenfunction corresponding to $\|C_\varphi\|^2$, in other words a function on which C_φ attains its norm, that vanishes at 0. Therefore $g_1 = (g_1(0)/g_2(0))\, g_2$, as we had hoped to show. □

2.2 The weighted Bergman spaces A^2_α

For $\alpha > -1$, the *weighted Bergman space* A^2_α is the Hilbert space consisting of all analytic functions f on $\mathbb{D}$ with

$$\|f\|_{2,\alpha}^2 = \int_{\mathbb{D}} |f(z)|^2 \,(\alpha+1)\left(1-|z|^2\right)^{\alpha} \frac{dA(z)}{\pi} < \infty,$$

where dA denotes area measure on $\mathbb{D}$. The inner product of two functions f and g in A^2_α is defined

$$\langle f, g\rangle_\alpha = \int_{\mathbb{D}} f(z)\overline{g(z)}(\alpha+1)\left(1-|z|^2\right)^{\alpha} \frac{dA(z)}{\pi}.$$

The space $A^2_0 = A^2$ is often referred to simply as the *(unweighted) Bergman space*. The inner product in A^2_α can also be expressed as a series; given $f(z) = \sum_{n=0}^\infty a_n z^n$ and $g(z) = \sum_{n=0}^\infty b_n z^n$ in A^2_α, we may write

$$\langle f, g \rangle_\alpha = \sum_{n=0}^\infty a_n \overline{b_n} \beta_\alpha(n),$$

where

$$\beta_\alpha(n) = \|z^n\|^2_{2,\alpha} = \frac{(\alpha+1)\Gamma(n+1)\Gamma(\alpha+1)}{\Gamma(n+\alpha+2)}.$$

(See, for example, Section 1 of [22].) In particular,

$$\beta_0(n) = \frac{1}{n+1} \text{ and } \beta_1(n) = \frac{2}{(n+1)(n+2)}.$$

Every weighted Bergman space A^2_α is a functional Hilbert space; the reproducing kernel functions for A^2_α may be written

$$K_w(z) = \left(\frac{1}{1-\overline{w}z}\right)^{\alpha+2},$$

where the power function is defined by taking the principal branch of the appropriate logarithm; note that

$$k_w(z) = \frac{K_w(z)}{\|K_w\|_{2,\alpha}} = \left(\frac{1-|w|^2}{(1-\overline{w}z)^2}\right)^{(\alpha+2)/2}.$$

As we can easily see, the kernel functions for the Hardy space H^2 have the form corresponding to those for the nonexistent "$\alpha = -1$" weighted Bergman space. This similarity will allow us to prove several results simultaneously for the Hardy space and the weighted Bergman spaces. For this reason, as a matter of convenience, we often write A^2_{-1} to denote the Hardy space H^2 (with the usual norm and inner product).

Composition operators have also received a good deal of attention in the context of the weighted Bergman spaces. Any analytic $\varphi : \mathbb{D} \to \mathbb{D}$ induces a bounded composition operator on every space A^2_α; in fact, for $\alpha > -1$, we have that

$$\left(\frac{1}{1-|\varphi(0)|^2}\right)^{(\alpha+2)/2} \leq \left\|C_\varphi : A^2_\alpha \to A^2_\alpha\right\| \leq \left(\frac{1+|\varphi(0)|}{1-|\varphi(0)|}\right)^{(\alpha+2)/2}. \tag{2.2}$$

(See, for example, Lemma 1.2.2 in [24].) Comparing with line (2.1), we see that these bounds are analogous to those for composition operators acting on $A^2_{-1} = H^2$. As in the case of the Hardy space, there are only a few examples of φ (with $\varphi(0) \neq 0$) for which we can calculate $\|C_\varphi : A^2_\alpha \to A^2_\alpha\|$. For instance, we know the norm of C_φ acting on the unweighted Bergman space A^2 when

(1) φ is an automorphism of $\mathbb{D}$; that is, $\varphi(z) = s\left(\frac{w-z}{1-\overline{w}z}\right)$ where $|w| < |s| = 1$,

(2) $\varphi(z) = az + b$ where $|a| + |b| \leq 1$,

(3) $\varphi(z) = \frac{(r+s)z+(1-s)}{r(1-s)z+(1+sr)}$ where $0 < s < 1$ and $-1/7 \leq r \leq 1$.

Case (2) was established by Hurst [15], who adapted Cowen's [9] proof for $C_\varphi : H^2 \to H^2$; in fact, Hurst determined the norm of such a composition operator acting on any of the weighted Bergman spaces. Case (3) follows from the cosubnormality results in Richman's thesis [24]. As one might expect, the maps in cases (1) and (3) induce extremally noncompact composition operators, as do the maps in case (2) when $|a| + |b| = 1$.

We conclude this section with a straightforward, but remarkably useful observation:

Observation 2.4 *Take $\alpha \geq -1$. Let λ be an eigenvalue for $C^*_\varphi C_\varphi : A^2_\alpha \to A^2_\alpha$ with a corresponding eigenfunction g; since C_φ fixes the constant function $K_0(z) = 1$, we see that*

$$\begin{aligned} g(\varphi(0)) &= \langle C_\varphi(g), K_0\rangle_\alpha = \langle C_\varphi(g), C_\varphi(K_0)\rangle_\alpha \\ &= \left\langle \left(C^*_\varphi C_\varphi\right)(g), K_0\right\rangle_\alpha = \lambda \langle g, K_0\rangle_\alpha = \lambda g(0). \end{aligned}$$

In particular, Proposition 1.2 dictates that

$$g(\varphi(0)) = \|C_\varphi\|^2 g(0)$$

whenever C_φ attains its norm on g.

2.3 Compactness and essential norms

While not much is known about the norm of C_φ, we do have a good deal of information about its essential norm. Joel Shapiro [26] obtained an explicit formula for $\|C_\varphi : H^2 \to H^2\|_e$; a similar formula was later discovered by Poggi-Corradini [22] for composition operators acting on $A_0^2 = A^2$ and A_1^2. This section includes the statements of these results, as well as several facts pertaining to the compactness of C_φ. We begin with the following lemma:

Lemma 2.5 *Take $\alpha \geq -1$. Let $\{f_n\}$ be a sequence of functions in A_α^2. The following three conditions are equivalent:*

- *(a) $\{f_n\}$ is bounded in norm and converges to 0 pointwise on $\mathbb{D}$.*
- *(b) $\{f_n\}$ is bounded in norm and converges to 0 uniformly on compact subsets of $\mathbb{D}$.*
- *(c) $\{f_n\}$ converges to 0 weakly in A_α^2.*

Proof. Corollary 1.3 in [11] shows that conditions (a) and (c) are equivalent; condition (b) certainly implies condition (a). Suppose then that $\{f_n\}$ is a sequence which converges to 0 pointwise on $\mathbb{D}$, with

$$\|f_n\|_{2,\alpha} \leq M < \infty$$

for all n. Let Δ be a compact subset of $\mathbb{D}$; then the quantity

$$H_\Delta = \sup_{w \in \Delta}\{|w|\}$$

is strictly less than 1. Hence, for every function f_n,

$$\begin{aligned} |f_n(w)| &= \langle f_n, K_w\rangle_\alpha \leq \|f_n\|_{2,\alpha} \|K_w\|_{2,\alpha} \\ &= \frac{\|f_n\|_{2,\alpha}}{\left(1-|w|^2\right)^{(\alpha+2)/2}} \leq \frac{M}{\left(1-(H_\Delta)^2\right)^{(\alpha+2)/2}} \end{aligned}$$

for all w in Δ. It follows from Montel's theorem that the sequence converges to 0 uniformly on every compact subset of $\mathbb{D}$. □

Using this characterization of weak convergence, we can easily prove a well-known result relating to the compactness of C_φ. The following proposition, stated for composition operators acting on H^2, first appeared in Schwartz's thesis [25]:

Proposition 2.6 *Take $\alpha \geq -1$. Let φ be an analytic self-map of $\mathbb{D}$. If the closure of the set $\varphi(\mathbb{D})$ is contained in $\mathbb{D}$, then the operator $C_\varphi : A^2_\alpha \to A^2_\alpha$ is compact.*

Proof. Let $\{f_n\}$ be a sequence in A^2_α that converges weakly to 0. Lemma 2.5 shows that the sequence converges to 0 uniformly on the closure of $\varphi(\mathbb{D})$. In other words, the functions $C_\varphi(f_n) = f_n \circ \varphi$ converge to 0 uniformly on $\mathbb{D}$, which implies that they converge to 0 in norm. Hence C_φ is completely continuous; since A^2_α is a Hilbert space, the operator must be compact. □

We now state Shapiro and Poggi-Corradini's essential norm formulae. For a real number $\rho \geq 0$, we define the *Nevanlinna-type counting function* $N_{\varphi,\rho}$ by the rule

$$N_{\varphi,\rho}(w) = \sum_{z\in\varphi^{-1}(\{w\})} (-\log|z|)^\rho$$

for any w in $\mathbb{D} \setminus \{\varphi(0)\}$. For α equal to -1, 0, or 1, Shapiro and Poggi-Corradini showed that

$$\left\|C_\varphi : A^2_\alpha \to A^2_\alpha\right\|_e^2 = \limsup_{|w|\uparrow 1} \frac{N_{\varphi,\alpha+2}(w)}{(-\log|w|)^{\alpha+2}}.$$

Whenever φ is univalent, this formula simply reduces to

$$\left\| C_\varphi : A^2_\alpha \to A^2_\alpha \right\|_e^2 = \limsup_{|w| \uparrow 1} \left(\frac{1 - |w|^2}{1 - |\varphi(w)|^2} \right)^{\alpha + 2} . \tag{2.3}$$

In the event that φ is univalent on a neighborhood of the closed unit disk, we have an even more attractive description of the essential norm. In this case, the Julia–Carathéodory theorem (see Theorem 2.44 in [11]) dictates that

$$\left\| C_\varphi : A^2_\alpha \to A^2_\alpha \right\|_e^2 = \max \left\{ |\varphi'(w)|^{-(\alpha+2)} : |w| = |\varphi(w)| = 1 \right\} , \tag{2.4}$$

where the maximum over an empty set is taken to be 0. In particular, such a φ induces a compact composition operator on one of these three spaces if and only if

$$\|\varphi\|_\infty = \sup_{w \in \mathbb{D}} \{ |\varphi(w)| \} < 1.$$

At this point, we mention a particularly helpful consequence of Shapiro and Poggi-Corradini's proofs. Both authors observe that

$$\|C_\varphi\|_e^2 \geq \limsup_{|w| \uparrow 1} \|C_\varphi(k_w)\|_{2,\alpha}^2 ,$$

which in turn they show to be greater than or equal to

$$\limsup_{|w| \uparrow 1} \frac{N_{\varphi, \alpha+2}(w)}{(-\log|w|)^{\alpha+2}} ,$$

the quantity they eventually prove to equal the square of the essential norm. Consequently

$$\|C_\varphi\|_e = \limsup_{|w| \uparrow 1} \|C_\varphi(k_w)\|_{2,\alpha} \tag{2.5}$$

whenever α equals -1, 0, or 1. This observation appears to have been made first in the context of H^2 by Cima and Matheson [7].

2.4 The quantities S_φ and S_φ^*

Having become acquainted with the Hardy space and the weighted Bergman spaces, we redirect our attention to the the quantities S_φ and S_φ^* (as defined in Section 1.3). Take $\alpha \geq -1$; throughout this section, unless otherwise stated, we assume that C_φ is a composition operator acting on A^2_α. Since every such space contains the constant functions, the set $N = \{w \in \mathbb{D} : f(w) = 0 \text{ for all } f \in A^2_\alpha\}$ is empty; therefore

$$S_\varphi = \sup_{w\in\mathbb{D}} \left\{ \|C_\varphi(k_w)\|_{2,\alpha} \right\} = \sup_{w\in\mathbb{D}} \left\{ \frac{\|C_\varphi(K_w)\|_{2,\alpha}}{\|K_w\|_{2,\alpha}} \right\}$$

and

$$\begin{aligned} S_\varphi^* &= \sup_{w\in\mathbb{D}} \left\{ \left\|C_\varphi^*(k_w)\right\|_{2,\alpha} \right\} = \sup_{w\in\mathbb{D}} \left\{ \frac{\left\|C_\varphi^*(K_w)\right\|_{2,\alpha}}{\|K_w\|_{2,\alpha}} \right\} \\ &= \sup_{w\in\mathbb{D}} \left\{ \frac{\left\|K_{\varphi(w)}\right\|_{2,\alpha}}{\|K_w\|_{2,\alpha}} \right\} = \sup_{w\in\mathbb{D}} \left\{ \left(\frac{1-|w|^2}{1-|\varphi(w)|^2} \right)^{(\alpha+2)/2} \right\}. \end{aligned}$$

It follows from Proposition 1.4 that $S_\varphi^* \leq S_\varphi$.

The next few remarks simplify our discussion of S_φ and S_φ^*.

Lemma 2.7 *Suppose that $\{w_j\}$ is a sequence of points converging to w in $\mathbb{D}$; then the normalized kernel functions $\{k_{w_j}\}$ converge to k_w in the norm of A^2_α.*

Proof. Simply note that

$$\begin{aligned} \left\|k_{w_j} - k_w\right\|^2_{2,\alpha} &= \left\langle k_{w_j} - k_w, k_{w_j} - k_w \right\rangle_\alpha \\ &= \left\|k_{w_j}\right\|^2_{2,\alpha} - 2\operatorname{Re}\left\langle k_w, k_{w_j}\right\rangle_\alpha + \|k_w\|^2_{2,\alpha} \\ &= 2 - 2\operatorname{Re}\left(\frac{\left\langle k_w, K_{w_j}\right\rangle_\alpha}{\left\|K_{w_j}\right\|_{2,\alpha}} \right) \\ &= 2 - 2\operatorname{Re}\left(k_w(w_j)\left(1-|w_j|^2\right)^{(\alpha+2)/2} \right). \end{aligned}$$

Since k_w is a continuous function,

$$\lim_{j\to\infty} k_w(w_j)\left(1-|w_j|^2\right)^{(\alpha+2)/2} = k_w(w)\left(1-|w|^2\right)^{(\alpha+2)/2} = 1,$$

and our claim follows. □

Since C_φ and C_φ^* are both bounded operators, we obtain the following characterization of S_φ and S_φ^*:

Observation 2.8 *Either* $S_\varphi = \|C_\varphi(k_w)\|_{2,\alpha}$ *for a particular* w *in* $\mathbb{D}$ *or*

$$S_\varphi = \limsup_{|w|\uparrow 1} \|C_\varphi(k_w)\|_{2,\alpha}.$$

Likewise, either $S_\varphi^* = \left\|C_\varphi^*(k_w)\right\|_{2,\alpha}$ *for some* w *in* $\mathbb{D}$ *or*

$$S_\varphi^* = \limsup_{|w|\uparrow 1} \left\|C_\varphi^*(k_w)\right\|_{2,\alpha} = \limsup_{|w|\uparrow 1} \left(\frac{1-|w|^2}{1-|\varphi(w)|^2}\right)^{(\alpha+2)/2}.$$

One of our main objectives is to determine the conditions under which either S_φ or S_φ^* equals $\|C_\varphi\|$. (If $S_\varphi^* = \|C_\varphi\|$, then of course $S_\varphi = \|C_\varphi\|$ as well.) Bourdon and Retsek [6] considered this question for composition operators acting on H^2; much of this section is a generalization of the material from Section 3 of their paper.

If $\varphi(0) = 0$, then this question does not demand a great deal of attention. In this case, expressions (2.1) and (2.2) dictate that $\|C_\varphi : A_\alpha^2 \to A_\alpha^2\| = 1$. Therefore

$$\|C_\varphi\| = \|k_0\|_{2,\alpha} = \left\|k_{\varphi(0)}\right\|_{2,\alpha} = \left\|C_\varphi^*(k_0)\right\|_{2,\alpha},$$

so that $S_\varphi^* = S_\varphi = \|C_\varphi\|$. Hence we need only consider the case where $\varphi(0) \neq 0$.

Our next lemma pertains to the quantity S_φ^*.

Lemma 2.9 *Suppose that* $\varphi : \mathbb{D} \to \mathbb{D}$ *is an analytic map with* $\varphi(0) \neq 0$. *If the operator* $C_\varphi^* : A_\alpha^2 \to A_\alpha^2$ *attains its norm on a kernel function, then* φ *must have the form* $\varphi(z) = az + b$.

Proof. Suppose that C_φ^* attains its norm on the kernel function K_w. Proposition 1.2 dictates that

$$\|C_\varphi\|^2 K_w = \left(C_\varphi C_\varphi^*\right)(K_w) = C_\varphi\left(K_{\varphi(w)}\right).$$

Hence

$$\frac{\|C_\varphi\|^2}{(1-\overline{w}z)^{\alpha+2}} = \left(\frac{1}{1-\overline{\varphi(w)}\varphi(z)}\right)^{\alpha+2}$$

for all z in $\mathbb{D}$. Thus it must be the case that

$$\frac{1-\overline{\varphi(w)}\varphi(z)}{1-\overline{w}z} = c \tag{2.6}$$

for some constant c. If $\varphi(w)$ happened to equal 0, then w would also be 0, contradicting the assumption that $\varphi(0) \neq 0$. Hence $\varphi(w) \neq 0$, so we can solve (2.6) explicitly for $\varphi(z)$ to show that it has the desired form. □

In light of this fact, it is not difficult to prove the natural generalization of Proposition 3.4 in [6]:

Proposition 2.10 *Let $\varphi : \mathbb{D} \to \mathbb{D}$ be an analytic map with $\varphi(0) \neq 0$ and which does not have the form $\varphi(z) = az + b$; consider the operator $C_\varphi : A^2_\alpha \to A^2_\alpha$. If the quantity S_φ^* equals $\|C_\varphi\|$, then $\|C_\varphi\| = \|C_\varphi\|_e$.*

Proof. The normalized kernel functions k_w are bounded in norm and converge pointwise to 0 as $|w|$ approaches 1; thus Lemma 2.5 dictates that the k_w converge weakly to 0 as $|w|$ tends to 1. Let K be a compact operator on A^2_α; observe that

$$\begin{aligned}
\left\|C_\varphi^* - K\right\| &\geq \limsup_{|w|\uparrow 1} \left\|\left(C_\varphi^* - K\right)(k_w)\right\|_{2,\alpha} \\
&\geq \limsup_{|w|\uparrow 1} \left(\left\|C_\varphi^*(k_w)\right\|_{2,\alpha} - \|K(k_w)\|_{2,\alpha}\right) \\
&= \limsup_{|w|\uparrow 1} \left\|C_\varphi^*(k_w)\right\|_{2,\alpha}.
\end{aligned}$$

Taking the infimum over the set of compact operators, we see that

$$\left\|C_\varphi\right\|_e = \left\|C_\varphi^*\right\|_e \geq \limsup_{|w|\uparrow 1} \left\|C_\varphi^*(k_w)\right\|_{2,\alpha}.$$

Lemma 2.9 tells us that $\|C_\varphi\| > \left\|C_\varphi^*(k_w)\right\|_{2,\alpha}$ for every w in $\mathbb{D}$. Therefore, if $\|C_\varphi\| = S_\varphi^*$, Observation 2.8 shows that

$$\|C_\varphi\| = S_\varphi^* = \limsup_{|w|\uparrow 1} \left\|C_\varphi^*(k_w)\right\|_{2,\alpha},$$

which implies that $\|C_\varphi\| = \left\|C_\varphi\right\|_e$. □

Bourdon and Retsek [6], in fact, proved that $S_\varphi^* = S_\varphi = \|C_\varphi\|$ whenever φ has the form $\varphi(z) = az + b$. Their proof, which is stated for composition operators acting on H^2, can be easily adapted to the weighted Bergman spaces A_α^2 for $\alpha > -1$.

Proposition 2.10 demonstrates how uncommon it is for S_φ^* to equal $\|C_\varphi\|$. The following result, which is specific to H^2, further illustrates this point:

Proposition 2.11 *Suppose that $\varphi : \mathbb{D} \to \mathbb{D}$ can be written as the composition $\mu \circ \nu$, where μ is an analytic self-map of $\mathbb{D}$ and ν is a non-univalent inner function. Consider the operator $C_\varphi : H^2 \to H^2$; if $\varphi(0) \neq 0$, then $S_\varphi^* < \|C_\varphi\|$.*

This proposition is a generalization of Theorem 3.5 in [6], which gives the result in the case where $\mu(z) = z$. The main idea of our proof is derived from Theorem 7.9 in [5]. The following argument requires a certain amount of geometric function theory; we refer the reader to Section 2.3 of [11] for the prerequisite material.

Proof of Proposition 2.11. Without loss of generality, we may assume that the inner function ν has the property that $\nu(0) = 0$. If $\nu(0)$ were nonzero, we could introduce the involutive automorphism

$$b(z) = \frac{\nu(0) - z}{1 - \overline{\nu(0)}z},$$

observing that $\varphi = \mu \circ \nu = (\mu \circ b) \circ (b \circ \nu)$. The map $\mu \circ b$ is an analytic self-map of the disk; $b \circ \nu$ is a non-univalent inner function that fixes the origin.

Since φ is not univalent, it cannot have the form $\varphi(z) = az + b$. Therefore, if S^*_φ were to equal $\left\|C^*_\varphi(k_w)\right\|_2$ for some particular w, Lemma 2.9 would dictate that $S^*_\varphi < \|C_\varphi\|$. Suppose then that $S^*_\varphi = \limsup_{|w|\uparrow 1} \left\|C^*_\varphi(k_w)\right\|_2$ (which means that the limit superior is positive). In this case, Proposition 2.46 in [11] shows that there is some ζ on $\partial\mathbb{D}$ such that

$$S^*_\varphi = \limsup_{|w|\uparrow 1} \sqrt{\frac{1 - |w|^2}{1 - |\varphi(w)|^2}} = \limsup_{|w|\uparrow 1} \sqrt{\frac{1 - |w|}{1 - |\varphi(w)|}} = |\varphi'(\zeta)|^{-1/2}.$$

Here $\varphi'(\zeta)$ denotes the (finite) angular derivative of φ at ζ.

Our first goal is to show that ν also has finite angular derivative at the point ζ. Since φ has finite angular derivative at ζ, it follows that

$$\lim_{r\uparrow 1} |\mu(\nu(r\zeta))| = \lim_{r\uparrow 1} |\varphi(r\zeta)| = 1,$$

and hence

$$\lim_{r\uparrow 1} |\nu(r\zeta)| = 1.$$

In other words, there is at least one sequence $\{w_n\}$ (which converges to ζ nontangentially) such that $\lim_{n\to\infty} |\nu(w_n)| = 1$. The Julia–Carathéodory theorem (see Theorem 2.44 in [11]) dictates that there is a sequence of points $\{z_n\}$, converging to ζ in a nontangential approach region, such that

$$\lim_{n\to\infty} \frac{1 - |\nu(z_n)|}{1 - |z_n|} = \liminf_{z\to\zeta} \frac{1 - |\nu(z)|}{1 - |z|}.$$

If the quantity

$$\limsup_{n\to\infty} |\nu(z_n)|$$

were strictly less than 1, then

$$\liminf_{z\to\zeta} \frac{1-|\nu(z)|}{1-|z|} \tag{2.7}$$

would be infinite. Since (2.7) is always less than or equal to ∞, we may assume that $\limsup_{n\to\infty} |\nu(z_n)| = 1$; passing to a subsequence, we assume further that $\{\nu(z_n)\}$ converges to a point ω on $\partial\mathbb{D}$. Note that

$$\liminf_{n\to\infty} \frac{1-|\mu(\nu(z_n))|}{1-|\nu(z_n)|} \geq \liminf_{z\to\omega} \frac{1-|\mu(z)|}{1-|z|},$$

a quantity which is always greater than 0. Since $|\nu(z)| < 1$ whenever $|z| < 1$, we may write

$$\frac{1-|\varphi(z)|}{1-|z|} = \frac{1-|\mu(\nu(z))|}{1-|\nu(z)|} \cdot \frac{1-|\nu(z)|}{1-|z|}$$

for all z in $\mathbb{D}$. Therefore

$$\liminf_{n\to\infty} \frac{1-|\varphi(z_n)|}{1-|z_n|} \geq \liminf_{n\to\infty} \frac{1-|\mu(\nu(z_n))|}{1-|\nu(z_n)|} \cdot \lim_{n\to\infty} \frac{1-|\nu(z_n)|}{1-|z_n|}.$$

The Julia–Carathéodory theorem shows that

$$\liminf_{n\to\infty} \frac{1-|\varphi(z_n)|}{1-|z_n|} = \lim_{n\to\infty} \frac{1-|\varphi(z_n)|}{1-|z_n|} = |\varphi'(\zeta)| < \infty.$$

Consequently

$$\liminf_{z\to\zeta} \frac{1-|\nu(z)|}{1-|z|} = \lim_{n\to\infty} \frac{1-|\nu(z_n)|}{1-|z_n|} < \infty,$$

from which it follows that ν has finite angular derivative at ζ.

Because of its finite angular derivative, ν must have radial limit of modulus 1 at ζ, a point on $\partial\mathbb{D}$ we denote $\nu(\zeta)$. Our next task is to show that μ has finite angular derivative at $\nu(\zeta)$. Since both φ and ν have finite angular derivative at ζ, the Julia–Carathéodory theorem dictates that

$$\begin{aligned}\liminf_{z\to\nu(\zeta)} \frac{1-|\mu(z)|}{1-|z|} &\le \limsup_{r\uparrow 1} \frac{1-|\mu(\nu(rz))|}{1-|\nu(rz)|}\\ &\le \limsup_{r\uparrow 1} \frac{1-|\varphi(rz)|}{1-|rz|}\cdot \limsup_{r\uparrow 1} \frac{1-|rz|}{1-|\nu(rz)|}\\ &= \frac{|\varphi'(\zeta)|}{|\nu'(\zeta)|} < \infty.\end{aligned}$$

Hence μ does indeed have finite angular derivative at the point $\nu(\zeta)$, with

$$|\mu'(\nu(\zeta))| \le \frac{|\varphi'(\zeta)|}{|\nu'(\zeta)|}. \tag{2.8}$$

We now return to the problem at hand. Since $\nu(0)=0$ and ν is not an automorphism, Lemma 7.33 in [11] shows that

$$\liminf_{z\to\zeta} \frac{1-|\nu(z)|}{1-|z|} > 1.$$

Appealing once more to the Julia–Carathéodory theorem, we see that that $|\nu'(\zeta)|>1$. Thus line (2.8) shows that

$$|\mu'(\nu(\zeta))| \le \frac{|\varphi'(\zeta)|}{|\nu'(\zeta)|} < |\varphi'(\zeta)|.$$

Consequently

$$\begin{aligned}|\varphi'(\zeta)|^{-1/2} &< |\mu'(\nu(\zeta))|^{-1/2} \le \limsup_{|w|\uparrow 1} \sqrt{\frac{1-|w|}{1-|\mu(w)|}}\\ &= \limsup_{|w|\uparrow 1} \sqrt{\frac{1-|w|^2}{1-|\mu(w)|^2}} = \limsup_{|w|\uparrow 1} \left\|C_\mu^*(k_w)\right\|_2.\end{aligned}$$

In the proof of Proposition 2.10, we noted that

$$\limsup_{|w|\uparrow 1} \left\|C_\mu^*(k_w)\right\|_2 \le \|C_\mu\|_e.$$

Since ν is an inner function that fixes the origin, Proposition 2.1 dictates that C_ν is an isometry of H^2. Therefore

$$\|C_\varphi\|_e = \|C_{\mu\circ\nu}\|_e = \|C_\nu C_\mu\|_e = \|C_\mu\|_e .$$

Combining these observations, we see that

$$S^*_\varphi < \|C_\varphi\|_e \leq \|C_\varphi\|,$$

as we had hoped to show. □

In particular, this result applies in the case where $\nu(z) = z^m$ for some integer $m \geq 2$, so that $\varphi(z) = \mu(z^m)$.

It is much more difficult to obtain information about the quantity S_φ. Most of our discussion of S_φ will be specific to composition operators $C_\varphi : H^2 \to H^2$, where $\varphi : \mathbb{D} \to \mathbb{D}$ is a linear fractional map (see Section 3.3). We can, though, obtain one general lemma relating to S_φ (Lemma 2.13); our proof depends on Lemma 2.12, which pertains to the lower bounds in expressions (2.1) and (2.2). Lemma 2.12, stated in the context of the Hardy space, appears with a different proof in a current paper of David Pakorny and Jonathan Shapiro [21].

Lemma 2.12 *If $\varphi : \mathbb{D} \to \mathbb{D}$ is a nonconstant analytic map with $\varphi(0) \neq 0$, then*

$$\left\|C_\varphi : A^2_\alpha \to A^2_\alpha\right\| > \left(\frac{1}{1-|\varphi(0)|^2}\right)^{(\alpha+2)/2} .$$

Proof. Our hypotheses guarantee that the function $k_{\varphi(0)} \circ \varphi$ is nonconstant; that is, it is not a scalar multiple of the kernel function $K_0(z) = 1$. Since K_0 has norm 1, we see that

$$\begin{aligned} \left(\frac{1}{1-|\varphi(0)|^2}\right)^{(\alpha+2)/2} &= \left|\left(k_{\varphi(0)} \circ \varphi\right)(0)\right| = \left\langle C_\varphi\left(k_{\varphi(0)}\right), K_0\right\rangle_\alpha \\ &< \left\|C_\varphi\left(k_{\varphi(0)}\right)\right\|_{2,\alpha} \|K_0\|_{2,\alpha} \leq \|C_\varphi\|, \end{aligned}$$

as we had hoped to show. ☐

Lemma 2.13 *Suppose that* $\varphi : \mathbb{D} \to \mathbb{D}$ *is a nonconstant analytic map with* $\varphi(0) \neq 0$. *If the operator* $C_\varphi : A^2_\alpha \to A^2_\alpha$ *attains its norm on a kernel function* K_w*, then* $|w| > |\varphi(0)|$.

Proof. Suppose that C_φ attains its norm on K_w; then K_w is an eigenfunction for $C^*_\varphi C_\varphi$ corresponding to $\|C_\varphi\|^2$. Appealing to Observation 2.4, we see that

$$\frac{1}{1-\overline{w}\varphi(0)} = K_w(\varphi(0)) = \|C_\varphi\|^2 K_w(0) = \|C_\varphi\|^2 .$$

It follows from Lemma 2.12 that

$$\frac{1}{1-\overline{w}\varphi(0)} > \frac{1}{1-|\varphi(0)|^2},$$

meaning that $|w| > |\varphi(0)|$. ☐

We end this section with a restatement of Observation 2.8, valid for α equal to -1, 0, or 1. Recalling equation (2.5) and the univalent version of the essential norm formula (2.3), we obtain the following characterization of the quantities S_φ and S^*_φ:

Observation 2.14 *Consider the composition operator* C_φ *acting on* A^2_α*, where* α *equals* -1, 0, *or* 1. *Either* $S_\varphi = \|C_\varphi(k_w)\|_{2,\alpha}$ *for a particular* w *in* $\mathbb{D}$ *or* $S_\varphi = \|C_\varphi\|_e$. *Similarly, if* φ *is univalent, then either* $S^*_\varphi = \left\|C^*_\varphi(k_w)\right\|_{2,\alpha}$ *for some* w *in* $\mathbb{D}$ *or* $S^*_\varphi = \|C_\varphi\|_e$.

2.5 Multiplication operators

In accordance with standard usage, we write H^∞ to denote the Banach space consisting of all bounded analytic function on $\mathbb{D}$, under the norm

$$\|f\|_\infty = \sup_{w\in\mathbb{D}}\{|f(w)|\}.$$

For an element γ of H^∞, we define the *multiplication operator* M_γ by the rule

$$(M_\gamma(f))(z) = \gamma(z)f(z).$$

It is not difficult to show that M_γ is bounded on each space A^2_α, with

$$\left\|M_\gamma : A^2_\alpha \to A^2_\alpha\right\| = \|\gamma\|_\infty$$

for every $\alpha \geq -1$. Multiplication operators belong to a larger class of operators on A^2_α, often called the *Toeplitz operators*.

The basic properties of multiplication operators are easy to determine. For example, if γ and η are both elements of H^∞, then

$$M_{a\gamma+b\eta} = aM_\gamma + bM_\eta$$

for any scalars a and b. The adjoint of a multiplication operator also behaves nicely with respect to the reproducing kernel functions of A^2_α. Observe that

$$\left\langle f, M^*_\gamma K_w\right\rangle_\alpha = \langle M_\gamma(f), K_w\rangle_\alpha = \gamma(w)f(w) = \langle\gamma(w)f, K_w\rangle_\alpha = \left\langle f, \overline{\gamma(w)}K_w\right\rangle_\alpha$$

for every f in A^2_α. Therefore $M^*_\gamma(K_w) = \overline{\gamma(w)}K_w$ for every kernel function K_w.

Abusing notation somewhat, we write M_z to denote the operator M_γ with $\gamma(z) = z$. It is particularly easy to describe the adjoint of the operator M_z acting on the Hardy space H^2. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be an element of H^2; observe that

$$\begin{aligned}(M_z^*(f))(w) &= \langle M_z^*(f), K_w\rangle = \langle f, M_z(K_w)\rangle \\ &= \left\langle \sum_{n=0}^{\infty} a_n z^n, \sum_{n=0}^{\infty} \overline{w}^n z^{n+1} \right\rangle = \sum_{n=1}^{\infty} a_n w^{n-1},\end{aligned}$$

which equals $f'(0)$ when $w = 0$ and equals

$$\frac{f(w) - f(0)}{w}$$

for every w in $\mathbb{D} \setminus \{0\}$. This fact will prove particularly useful in Chapter 3, when we consider composition operators with linear fractional symbol.

2.6 Functional Banach spaces

Let $\mathcal{X}$ be a nontrivial Banach space whose elements are functions on a set U. As one would expect, $\mathcal{X}$ is called a *functional Banach space* if every point-evaluation functional is bounded on $\mathcal{X}$. A *Banach space of analytic functions* is simply a functional Banach space on an open subset U of $\mathbb{C}^n$ whose elements are analytic functions. It is perfectly reasonable to consider a composition operator acting on a functional Banach space; the proof of Theorem 1.1 is also valid in this context.

The Hilbert spaces H^2 and A^2_α, in fact, belong to a larger class of Banach spaces. For $1 \le p < \infty$, the *Hardy space* H^p is the Banach space consisting of all analytic functions f on $\mathbb{D}$ with

$$\|f\|_p^p = \lim_{r\uparrow 1} \int_0^{2\pi} \left|f\left(re^{i\theta}\right)\right|^p \frac{d\theta}{2\pi} = \int_0^{2\pi} \left|f\left(e^{i\theta}\right)\right|^p \frac{d\theta}{2\pi} < \infty.$$

Similarly, for $\alpha > -1$ and $1 \le p < \infty$, the *weighted Bergman space* A^p_α consists of those analytic functions f with

$$\|f\|_{p,\alpha}^p = \int_{\mathbb{D}} |f(z)|^p (\alpha+1)\left(1-|z|^2\right)^\alpha \frac{dA(z)}{\pi} < \infty.$$

Each of these spaces is a Banach space of analytic functions on $\mathbb{D}$. Extending our previous convention, we often write A^p_{-1} to denote the Hardy space H^p.

Any analytic $\varphi : \mathbb{D} \to \mathbb{D}$ induces a bounded composition operator on each weighted Bergman space. In fact,

$$\left(\frac{1}{1-|\varphi(0)|^2}\right)^{(\alpha+2)/p} \leq \|C_\varphi : A^p_\alpha \to A^p_\alpha\| \leq \left(\frac{1+|\varphi(0)|}{1-|\varphi(0)|}\right)^{(\alpha+2)/p}$$

for any $\alpha \geq -1$ and $1 \leq p < \infty$. In principle, for a fixed α, the calculation of $\|C_\varphi : A^p_\alpha \to A^p_\alpha\|$ might constitute a substantially different problem for each value of p. When dealing with the Hardy spaces, however, we need only consider C_φ acting on the Hilbert space H^2. In particular, we have the following result:

Proposition 2.15 *Take* $1 \leq p < \infty$. *For any analytic map* $\varphi : \mathbb{D} \to \mathbb{D}$,

$$\|C_\varphi : H^p \to H^p\|^p = \left\|C_\varphi : H^2 \to H^2\right\|^2 .$$

This proposition has been described as a “folk result”; it has been commonly known for several years (see, for example, [16]), but its original source remains obscure.

Proof of Proposition 2.15. Let $\{f_n\}$ be a sequence of unit vectors in H^p such that

$$\lim_{n\to\infty} \|C_\varphi(f_n)\|_p = \|C_\varphi : H^p \to H^p\| .$$

For each n, let B_n denote the Blaschke product whose zero-set is identical to that of f_n. Every Blaschke product is an isometric zero-divisor for H^p; in other words, each function

$$g_n = \frac{f_n}{B_n}$$

is also a unit vector in H^p. Moreover, since $|B_n(z)| < 1$ for all z in $\mathbb{D}$,

$$\|C_\varphi(g_n)\|_p = \left\|\frac{f_n \circ \varphi}{B_n \circ \varphi}\right\|_p \geq \|f_n \circ \varphi\|_p = \|C_\varphi(f_n)\|_p , \tag{2.9}$$

which means that

$$\lim_{n\to\infty} \|C_\varphi(g_n)\|_p = \|C_\varphi : H^p \to H^p\| \, .$$

Each g_n, being a non-vanishing analytic function on $\mathbb{D}$, has an analytic logarithm. Hence there exist analytic functions $(g_n)^{p/2}$, with

$$\left\|(g_n)^{p/2}\right\|_2^2 = \int_0^{2\pi} \left|\left(g_n\left(e^{i\theta}\right)\right)^{p/2}\right|^2 \frac{d\theta}{2\pi} = \int_0^{2\pi} \left|g_n\left(e^{i\theta}\right)\right|^p \frac{d\theta}{2\pi} = \|g_n\|_p^p = 1.$$

Therefore

$$\begin{aligned}
\left\|C_\varphi : H^2 \to H^2\right\|^2 &\geq \limsup_{n\to\infty} \left\|C_\varphi\left((g_n)^{p/2}\right)\right\|_2^2 \\
&= \limsup_{n\to\infty} \int_0^{2\pi} \left|\left(g_n\left(\varphi\left(e^{i\theta}\right)\right)\right)^{p/2}\right|^2 \frac{d\theta}{2\pi} \\
&= \limsup_{n\to\infty} \int_0^{2\pi} \left|g_n\left(\varphi\left(e^{i\theta}\right)\right)\right|^p \frac{d\theta}{2\pi} \\
&= \lim_{n\to\infty} \|C_\varphi(g_n)\|_p^p = \|C_\varphi : H^p \to H^p\|^p \, .
\end{aligned}$$

Switching the roles of p and 2, we obtain the reverse inequality. ☐

It is unknown whether the analogous result holds for a composition operator acting on the spaces A^p_α for a fixed $\alpha > -1$. The proof of Proposition 2.15 relies heavily on the fact that Blaschke products are isometric zero-divisors for H^p. It might seem natural, should we attempt to extend this proof to the weighted Bergman spaces, to substitute the canonical divisors of A^p_α for the Blaschke products; the canonical divisors, however, are often unbounded on $\mathbb{D}$, which means that line (2.9) is not necessarily valid in this context.

Chapter 3

Composition Operators with Linear Fractional Symbol

Throughout this chapter, we take

$$\varphi(z) = \frac{az+b}{cz+d} \tag{3.1}$$

to be a nonconstant linear fractional self-map of $\mathbb{D}$. We would like to obtain information about the norm of the corresponding composition operator, either on the Hardy space H^2 or the weighted Bergman spaces A^2_α. As we have already mentioned, our strategy involves investigating the spectrum of $C_\varphi^* C_\varphi$. Needless to say, this avenue of inquiry would be useless without a reasonably good representation for the adjoint of C_φ. Our results depend on the fact that, for this particular class of maps, we do have an explicit formula for C_φ^* (Proposition 3.3). Appealing to this formula, we establish a concrete representation for the operator $C_\varphi^* C_\varphi : H^2 \to H^2$, which we use to determine the conditions under which either S_φ or S_φ^* equals $\|C_\varphi : H^2 \to H^2\|$ (Theorem 3.7).

3.1 Cowen's adjoint formula

In this section, we derive a formula for C_φ^*. We begin with the following lemma:

Lemma 3.1 *Let $\varphi : \mathbb{D} \to \mathbb{D}$ be a nonconstant linear fractional map, as in (3.1); then the map*

$$\sigma(z) = \frac{\overline{a}z - \overline{c}}{-\overline{b}z + \overline{d}} \tag{3.2}$$

is also a self-map of $\mathbb{D}$; furthermore, both

$$\gamma(z) = \frac{1}{-\overline{b}z + \overline{d}}$$

and

$$\eta(z) = cz + d$$

are bounded, non-vanishing analytic functions on $\mathbb{D}$.

Proof. For the moment, let us view the nonconstant linear fractional maps as being bijections of the Riemann sphere. Define the set

$$\check{\mathbb{D}} = \{z : |z| > 1\} \cup \{\infty\}.$$

Since φ is a self-map of $\mathbb{D}$, the inverse map φ^{-1} must take $\check{\mathbb{D}}$ into $\check{\mathbb{D}}$; if this were not the case, then $\varphi^{-1}(\check{\mathbb{D}})$ would contain a point w in $\mathbb{D}$, in which case $\varphi(w)$ would belong to $\check{\mathbb{D}}$. A straightforward calculation shows that

$$\sigma = \rho \circ \varphi^{-1} \circ \rho,$$

where $\rho(z) = 1/\overline{z}$. Note that ρ maps $\mathbb{D}$ into $\check{\mathbb{D}}$, that φ^{-1} maps $\check{\mathbb{D}}$ into $\check{\mathbb{D}}$, and that ρ maps $\check{\mathbb{D}}$ into $\mathbb{D}$; therefore σ takes $\mathbb{D}$ into $\mathbb{D}$, as we had hoped to show.

Because σ is a self-map of $\mathbb{D}$, it cannot have a singularity in the closed unit disk. Since γ and σ have the same denominator, it follows that γ is bounded on $\mathbb{D}$; it is evident that γ does not vanish on $\mathbb{D}$. The function η is bounded on $\mathbb{D}$; observe that η is the denominator of φ, which implies that η does not vanish on $\mathbb{D}$. $\square$

Since ρ is the identity map on $\partial\mathbb{D}$, we make the following observation, which will prove useful in later sections:

Observation 3.2 *Let $\varphi : \mathbb{D} \to \mathbb{D}$ be a nonconstant linear fractional map, as in (3.1), and let σ be defined as in (3.2). Suppose that ζ and ω are points on $\partial\mathbb{D}$; then $\varphi(\zeta) = \omega$ if and only if $\sigma(\omega) = \zeta$.*

Having defined the functions σ, γ, and η, we turn our attention to the Hardy space and the weighted Bergman spaces. Fix a number $\alpha \geq -1$. It follows from Lemma 3.1 that the functions

$$\gamma_\alpha(z) = \gamma(z)^{\alpha+2} = \left(\frac{1}{-\bar{b}z + \bar{d}}\right)^{\alpha+2} \tag{3.3}$$

and

$$\eta_\alpha(z) = \eta(z)^{\alpha+2} = (cz + d)^{\alpha+2} \tag{3.4}$$

belong to H^∞. We write M_{γ_α} and to M_{η_α} denote the corresponding multiplication operators on A^2_α; these operators will appear in our formula for the adjoint of C_φ. The following proposition was originally proved by Cowen [9] for composition operators acting on H^2; Hurst [15] later extended the result to the weighted Bergman spaces:

Proposition 3.3 (Cowen's adjoint formula) *Take $\alpha \geq -1$. Let $\varphi : \mathbb{D} \to \mathbb{D}$ be a nonconstant linear fractional map, as in (3.1); let σ, γ_α, and η_α be defined as in (3.2), (3.3), and (3.4). The adjoint C^*_φ on A^2_α is given by the formula*

$$C^*_\varphi = M_{\gamma_\alpha} C_\sigma M^*_{\eta_\alpha}.$$

Proof. Recall that the reproducing kernel functions for A^2_α have the form

$$K_w(z) = \left(\frac{1}{1-\overline{w}z}\right)^{\alpha+2};$$

moreover, $C^*_\varphi(K_w) = K_{\varphi(w)}$ and $M^*_{\eta_\alpha}(K_w) = \overline{\eta_\alpha(w)}K_w$ for every w in $\mathbb{D}$. Observe that

$$\begin{aligned}
\left(\left(M_{\gamma_\alpha}C_\sigma M^*_{\eta_\alpha}\right)(K_w)\right)(z) &= \overline{\eta_\alpha(w)}\gamma_\alpha(z)(C_\sigma(K_w))(z)\\
&= \left(\overline{cw+d}\right)^{\alpha+2}\left(\frac{1}{-\overline{b}z+\overline{d}}\right)^{\alpha+2}\left(\frac{1}{1-\overline{w}\left(\frac{\overline{a}z-\overline{c}}{-\overline{b}z+\overline{d}}\right)}\right)^{\alpha+2}\\
&= \left(\frac{\overline{c}\overline{w}+\overline{d}}{-\overline{b}z+\overline{d}-\overline{w}\overline{a}z+\overline{w}\overline{c}}\right)^{\alpha+2}\\
&= \left(\frac{1}{1-\overline{\varphi(w)}z}\right)^{\alpha+2}\\
&= K_{\varphi(w)}(z) = \left(C^*_\varphi(K_w)\right)(z).
\end{aligned}$$

Therefore the operators C^*_φ and $M_{\gamma_\alpha}C_\sigma M^*_{\eta_\alpha}$ agree on every kernel function K_w; since the kernel functions span a dense subset of A^2_α, the two operators must be identical.

□

3.2 The operator $C^*_\varphi C_\varphi$

Our goal now is to use Cowen's adjoint formula to obtain a concrete representation for the operator $C^*_\varphi C_\varphi$. Our first observation pertains to maps of the form $\varphi(z) = az + b$, where $|a| + |b| \le 1$. Such a map, of course, may be written in linear fractional form, as in (3.1), with $c = 0$ and $d = 1$. Observe that $M^*_{\eta_\alpha} = M^*_1$; since multiplication by 1 is the identity operator on A^2_α, so too is its adjoint. Hence Proposition 3.3 dictates that, for any $\alpha \ge -1$, the adjoint $C^*_\varphi : A^2_\alpha \to A^2_\alpha$ may be written as the product $M_{\gamma_\alpha}C_\sigma$. Thus we obtain the following result:

Observation 3.4 *Consider the map* $\varphi(z) = az + b$ *where* $|a| + |b| \leq 1$. *For any* $\alpha \geq -1$, *the operator* $C_\varphi^* C_\varphi : A_\alpha^2 \to A_\alpha^2$ *has the form*

$$\begin{aligned} \left(\left(C_\varphi^* C_\varphi\right)(f)\right)(z) &= \left(\frac{1}{1-\overline{b}z}\right)^{\alpha+2} f(\varphi(\sigma(z))) \\ &= \left(\frac{1}{1-\overline{b}z}\right)^{\alpha+2} f(\tau(z)), \end{aligned}$$

where τ *denotes the composition* $\varphi \circ \sigma$.

We will make use of this representation in Section 4.2.

For the rest of this section, we restrict our attention to composition operators acting on H^2. As before, let $\varphi : \mathbb{D} \to \mathbb{D}$ be a nonconstant linear fractional map. The only piece of Cowen's adjoint formula that requires further discussion is the operator $M_{\eta_\alpha}^* = M_\eta^*$. It follows from our discussion in Section 2.5 that

$$\left(M_\eta^*(f)\right)(z) = \left(M_{cz+d}^*(f)\right)(z) = \left(\left(\overline{c}M_z^* + \overline{d}M_1^*\right)(f)\right)(z),$$

for every f in H^2, which equals

$$\overline{c}f'(0) + \overline{d}f(0)$$

when $z = 0$ and

$$\overline{c}\left(\frac{f(z) - f(0)}{z}\right) + \overline{d}f(z)$$

for all z in $\mathbb{D} \setminus \{0\}$.

This observation allows us to obtain a particularly useful representation for the operator $C_\varphi^* C_\varphi$. For any f in H^2, we see that

$$\begin{aligned} \left(\left(C_\varphi^* C_\varphi\right) f\right)(z) &= \left(\left(M_{\gamma_\alpha} C_\sigma M_{\eta_\alpha}^* C_\varphi\right)(f)\right)(z) \\ &= \left(\left(M_\gamma C_\sigma M_\eta^* C_\varphi\right)(f)\right)(z), \end{aligned}$$

which equals

$$\begin{aligned} &\gamma(z)\left(\overline{c}\left(\frac{f(\varphi(\sigma(z))) - f(\varphi(0))}{\sigma(z)}\right) + \overline{d}f(\varphi(\sigma(z)))\right) \\ = \quad &\frac{\overline{c}}{\overline{a}z - \overline{c}}\left[f(\varphi(\sigma(z))) - f(\varphi(0))\right] + \frac{\overline{d}}{-\overline{b}z + \overline{d}} f(\varphi(\sigma(z))) \end{aligned} \tag{3.5}$$

for all z in $\mathbb{D}$ not equal to $\sigma^{-1}(0) = \overline{c}/\overline{a}$. Performing the obvious algebraic manipulations, we arrive at the following result:

Observation 3.5 *Let $\varphi : \mathbb{D} \to \mathbb{D}$ be a nonconstant linear fractional map, as in (3.1). The operator $C_\varphi^* C_\varphi : H^2 \to H^2$ may be written*

$$\left(\left(C_\varphi^* C_\varphi\right) f\right)(z) = \psi(z) f(\tau(z)) + \chi(z) f(\varphi(0)), \tag{3.6}$$

where τ denotes the composition $\varphi \circ \sigma$ and

$$\psi(z) = \frac{\left(\overline{ad} - \overline{bc}\right) z}{\left(\overline{a}z - \overline{c}\right)\left(-\overline{b}z + \overline{d}\right)} \text{ and } \chi(z) = \frac{\overline{c}}{-\overline{a}z + \overline{c}}.$$

This representation holds for all z in $\mathbb{D} \setminus \{\sigma^{-1}(0)\}$.

The point $\sigma^{-1}(0) = \overline{c}/\overline{a}$ is something of a nuisance, but will not cause us any serious problems. For many choices of φ, this point is totally innocuous; it only lies in $\mathbb{D}$ if $|c| < |a|$. In the event that $\sigma^{-1}(0)$ does belong to $\mathbb{D}$, we can apply l'Hospital's rule to expression (3.5) to see that

$$\left(C_\varphi^* C_\varphi(f)\right)(\sigma^{-1}(0)) = \frac{\overline{c}}{\overline{a}} f'(\varphi(0))\tau'(\sigma^{-1}(0)) + \frac{\overline{ad}}{\overline{ad} - \overline{bc}} f(\varphi(0)). \tag{3.7}$$

3.3 The quantities S_φ and S_φ^*

In this section, we restrict our attention to composition operators acting on the Hardy space H^2. Applying Observation 3.5, we determine the conditions under which either $S_\varphi = \|C_\varphi\|$ or $S^*_\varphi = \|C_\varphi\|$ when $\varphi : \mathbb{D} \to \mathbb{D}$ is a linear fractional map. The following proposition constitutes the crux of our argument:

Proposition 3.6 *Let $\varphi : \mathbb{D} \to \mathbb{D}$ be a linear fractional map with $\varphi(0) \neq 0$ and which does not have the form $\varphi(z) = az + b$. For any point w in $\mathbb{D}$,*

$$\|C_\varphi\| > \|C_\varphi(k_w)\|_2 .$$

Proof. Suppose, to the contrary, that C_φ attains its norm on some normalized kernel function k_w; then K_w is an eigenfunction for $C^*_\varphi C_\varphi$. Consequently the subspace $\mathcal{K}_{\{w\}} = \{\alpha K_w : \alpha \in \mathbb{C}\}$ is invariant under $C^*_\varphi C_\varphi$. Since $C^*_\varphi C_\varphi$ is self-adjoint, the orthogonal complement $\mathcal{K}^\perp_{\{w\}} = \{f \in H^2 : f(w) = 0\}$ is also invariant under the operator; this observation will give rise to a contradiction.

Lemma 2.13 tells us that w cannot equal 0 or $\varphi(0)$. Suppose then that w is the problematic point $\sigma^{-1}(0) = \overline{c}/\overline{a}$. Referring to line (3.7), we see that

$$\left(C^*_\varphi C_\varphi(f)\right)(\sigma^{-1}(0)) = \frac{\overline{c}}{\overline{a}} f'(\varphi(0))\tau'(\sigma^{-1}(0)) + \frac{\overline{ad}}{\overline{ad} - \overline{bc}} f(\varphi(0)),$$

which must equal 0 for any f in $\mathcal{K}^\perp_{\{w\}}$. Consider the function $f_1(z) = (z-\varphi(0))(z-w)$ in $\mathcal{K}^\perp_{\{w\}}$. The assumption that $\varphi(z) \neq az+b$ guarantees that $c \neq 0$; since $f_1(\varphi(0)) = 0$ and $\tau = \varphi \circ \sigma$ is univalent, the term $f'_1(\varphi(0))$ must equal 0, which is not the case. Therefore w cannot equal $\sigma^{-1}(0)$. Hence equation (3.6) is valid at w, meaning that

$$0 = \left(C^*_\varphi C_\varphi(f)\right)(w) = \psi(w) f(\tau(w)) + \chi(w) f(\varphi(0))$$

for all f in $\mathcal{K}^\perp_{\{w\}}$.

Again consider the function f_1. Observe that $f_1(\varphi(0)) = 0$; since $w \neq 0$, the term $\psi(w)$ is nonzero. Hence $f_1(\tau(w)) = 0$, meaning that $\tau(w)$ equals either w or $\varphi(0)$. If $\tau(w) = \varphi(0)$, then $w = \sigma^{-1}(0)$, which is not the case; therefore $\tau(w) = w$. Now take

$f_2(z) = z - w$ in $\mathcal{K}_{\{w\}}^{\perp}$. Since $f_2(\tau(w)) = f_2(w) = 0$ and $\chi(w) = \bar{c}/(\bar{c} - \bar{a}w) \neq 0$, we see that $f_2(\varphi(0)) = 0$. Therefore $\varphi(0)$ must equal w, which is a contradiction. $\square$

Proposition 3.6 allows us easily to prove the main result of this section:

Theorem 3.7 *Let* $\varphi : \mathbb{D} \to \mathbb{D}$ *be a linear fractional map with* $\varphi(0) \neq 0$ *and which does not have the form* $\varphi(z) = az + b$. *Then* $S_\varphi = \|C_\varphi\|$ *if and only if* $\|C_\varphi\|_e = \|C_\varphi\|$*; likewise* $S_\varphi^* = \|C_\varphi\|$ *if and only if* $\|C_\varphi\|_e = \|C_\varphi\|$.

Proof. Observation 2.14 shows that $\|C_\varphi\|_e \le S_\varphi^* \le S_\varphi \le \|C_\varphi\|$ whenever φ is univalent; if $\|C_\varphi\|_e = \|C_\varphi\|$, then all of these quantities must be equal. On the other hand, suppose that $\|C_\varphi\|_e < \|C_\varphi\|$. Proposition 3.6 dictates that $\|C_\varphi(k_w)\|_2 < \|C_\varphi\|$ for all w in $\mathbb{D}$; it follows from Observation 2.14 that $S_\varphi < \|C_\varphi\|$. Since $S_\varphi^* \le S_\varphi$, our claim follows. $\square$

As a consequence of this theorem, we see that $S_\varphi = \|C_\varphi\|$ if and only if $S_\varphi^* = \|C_\varphi\|$. We should mention, though, that there are linear fractional φ such that $S_\varphi^* = S_\varphi < \|C_\varphi\|$; for example, Retsek [23] showed that the map $\varphi(z) = 4/(5 - z)$ has this property.

Theorem 3.7 no longer holds if we eliminate the hypothesis that φ be linear fractional. In light of Proposition 2.10, we see that our assertion for S_φ^* holds whenever φ is univalent (an observation also made by Retsek [23]). On the other hand, Bourdon and Retsek [6] proved that $S_\varphi^* < \|C_\varphi\|_e = \|C_\varphi\|$ whenever φ is a non-univalent inner function with $\varphi(0) \neq 0$ (see Proposition 2.11 above). Extremal noncompactness implies that $S_\varphi = \|C_\varphi\|$ for any φ. It is not difficult, however, to find further examples of analytic φ with $\varphi(0) \neq 0$ and $\|C_\varphi\|_e < S_\varphi = \|C_\varphi\|$. To that end, let ν

be an inner function that fixes the origin; then (by Proposition 2.1) the composition operator C_ν is an isometry of H^2. Hence, for any analytic $\varphi : \mathbb{D} \to \mathbb{D}$, the operator $C_{\varphi\circ\nu} = C_\nu C_\varphi$ has the same norm as C_φ; moreover, $S_{\varphi\circ\nu} = \|C_{\varphi\circ\nu}\|$ if and only if $S_\varphi = \|C_\varphi\|$. Consider the map $\varphi(z) = az + b$, where both a and b are nonzero and $|a| + |b| < 1$. We know that $S_\varphi = \|C_\varphi\|$, and that both of the operators C_φ and $C_{\varphi\circ\nu}$ are compact. Hence $\|C_{\varphi\circ\nu}\|_e = 0 < S_{\varphi\circ\nu} = \|C_{\varphi\circ\nu}\|$; in particular, this result holds if we take $\nu(z) = z^m$ for some integer $m \geq 1$, so that $(\varphi \circ \nu)(z) = az^m + b$.

Chapter 4

The Spectrum of a Weighted Composition Operator

Given an analytic map $\varphi : \mathbb{D} \to \mathbb{D}$ and a function ψ in H^∞, we define the *weighted composition operator* $W_{\varphi,\psi} : A^2_\alpha \to A^2_\alpha$ by the rule

$$\left(W_{\varphi,\psi}(f)\right)(z) = \psi(z) f(\varphi(z))$$

for all f in A^2_α and all z in $\mathbb{D}$. Such an operator, of course, is simply the product of the multiplication operator M_ψ with the composition operator C_φ. It seems plausible, in view of Cowen's adjoint formula, that information relating to the spectrum of $W_{\varphi,\psi}$ might be pertinent to our investigation of the spectrum of $C^*_\varphi C_\varphi$. Section 4.1 introduces a slightly more general class of operators, one that includes many of the weighted composition operators; we determine the spectrum of any such operator which happens to be compact. Section 4.2 offers a new proof, based on the work of the preceding section, of a previously known norm result for composition operators acting on A^2_α.

4.1 The spectrum

Let $\mathcal{X}$ be a Banach space of analytic functions on $\mathbb{D}$. Take $R_{\varphi,\psi}$ to be a bounded operator on $\mathcal{X}$ with the following properties:

> *There is a point* w_0 *in* $\mathbb{D}$*, an open subset* Δ *of* $\mathbb{D}$*, an analytic map* $\varphi : \mathbb{D} \to \mathbb{D}$*, and an analytic function* $\psi : \Delta \to \mathbb{C}$*, such that*
>
> - *The point* w_0 *belongs to* Δ*,*
> - *The map* φ *fixes the point* w_0*,*
> - $(R_{\varphi,\psi}(f))(z) = \psi(z)f(\varphi(z))$ *for all* f *in* $\mathcal{X}$ *and all* z *in* Δ*.*

To avoid several minor difficulties, we also assume that $\varphi'(w_0) \neq 0$.

The operator $R_{\varphi,\psi}$ behaves like a weighted composition operator in a neighborhood of w_0; one could reasonably describe such an operator as *locally resembling* a weighted composition operator on the set Δ. Having such a representation for $R_{\varphi,\psi}$, we write $\mathcal{M}$ to denote the set of all non-negative integers m such that there is some function x_m in $\mathcal{X}$ with a zero of order m at w_0. (If 0 belongs to $\mathcal{M}$, then $\mathcal{X}$ simply contains a function which does not vanish at w_0.) This set plays an important part in describing the spectrum of $R_{\varphi,\psi}$. We begin with the following observation:

Proposition 4.1 *Suppose that* φ *is not an automorphism of* $\mathbb{D}$ *and that* $\psi(w_0)$ *is nonzero. Let* m *be an element of* $\mathcal{M}$*; then the point* $\psi(w_0)\varphi'(w_0)^m$ *belongs to the spectrum of* $R_{\varphi,\psi}$*.*

Proof. Suppose that $\psi(w_0)\varphi'(w_0)^m$ belongs to the resolvent of $R_{\varphi,\psi}$; in other words, the operator $R_{\varphi,\psi} - \psi(w_0)\varphi'(w_0)^m$ is invertible. Hence there is some f in $\mathcal{X}$ such that

$$R_{\varphi,\psi}(f) - \psi(w_0)\varphi'(w_0)^m f = x_m,$$

where x_m is an element of $\mathcal{X}$ with a zero of order m at w_0. In other words,

$$\psi(z)f(\varphi(z)) - \psi(w_0)\varphi'(w_0)^m f(z) = x_m(z) \tag{4.1}$$

for all z in Δ.

First of all, suppose that $m = 0$. Since w_0 belongs to Δ, we have that

$$\begin{aligned} x_0(w_0) &= \psi(w_0)f(\varphi(w_0)) - \psi(w_0)f(w_0) \\ &= \psi(w_0)f(w_0) - \psi(w_0)f(w_0) = 0, \end{aligned}$$

which contradicts our assumption that $x_0(w_0) \neq 0$. Therefore $\psi(w_0) = \psi(w_0)\varphi'(w_0)^m$ must belong to the spectrum of $R_{\varphi,\psi}$.

Now suppose that $m \geq 1$. We claim that the jth derivative $f^{(j)}(w_0)$ must equal 0 for all $0 \leq j \leq m-1$, a fact we shall prove by induction. Observe that

$$\begin{aligned} 0 &= x_m(w_0) = \psi(w_0)f(\varphi(w_0)) - \psi(w_0)\varphi'(w_0)^m f(w_0) \\ &= \psi(w_0)\left(1 - \varphi'(w_0)^m\right) f(w_0). \end{aligned}$$

Since φ is not an automorphism of $\mathbb{D}$, it follows from the Schwarz lemma that $|\varphi'(w_0)| < 1$ (see, for example, p. 79 of [27]); therefore, since $\psi(w_0) \neq 0$, the term $f(w_0) = f^{(0)}(w_0)$ must equal 0. Now suppose that $f^{(k)}(w_0) = 0$ for all $0 \leq k \leq j-1$. Taking the jth derivative of both sides of (4.1), we see that

$$\left(\frac{d^j}{dz^j}[\psi(z)f(\varphi(z))]\bigg|_{z=w_0} \right) - \psi(w_0)\varphi'(w_0)^n f^{(j)}(w_0) = x_m^{(j)}(w_0) = 0.$$

The term

$$\frac{d^j}{dz^j}[\psi(z)f(\varphi(z))]\bigg|_{z=w_0}$$

consists of $\psi(w_0)\varphi'(w_0)^j f^{(j)}(w_0)$, plus the sum of various terms involving lower order derivatives of f at w_0. These extraneous terms all equal zero, which means that

$$\begin{aligned} 0 &= x_m^{(j)}(w_0) = \psi(w_0)\varphi'(w_0)^j f^{(j)}(w_0) - \psi(w_0)\varphi'(w_0)^m f^{(j)}(w_0) \\ &= \psi(w_0)\varphi'(w_0)^j \left(1 - \varphi'(w_0)^{m-j}\right) f^{(j)}(w_0). \end{aligned}$$

Therefore $f^{(j)}(w_0) = 0$ for all $0 \le j \le m-1$, as we had hoped to show.

Consequently we see that

$$\begin{aligned} 0 &\neq x_m^{(m)}(w_0) = \left(\frac{d^m}{dz^m} [\psi(z) f(\varphi(z))] \bigg|_{z=w_0} \right) - \psi(w_0)\varphi'(w_0)^m f^{(m)}(w_0) \\ &= \psi(w_0)\varphi'(w_0)^m f^{(m)}(w_0) - \psi(w_0)\varphi'(w_0)^m f^{(m)}(w_0) = 0, \end{aligned}$$

thereby obtaining a contradiction. In other words, $\psi(w_0)\varphi'(w_0)^m$ belongs to the spectrum of $R_{\varphi,\psi}$. □

At this point, we turn our attention specifically to the eigenvalues of $R_{\varphi,\psi}$.

Proposition 4.2 *Let λ be an eigenvalue of $R_{\varphi,\psi}$ with a corresponding eigenfunction g; then $\lambda = \psi(w_0)\varphi'(w_0)^m$, where m denotes the order of the zero of g at the point w_0.*

Proof. Observe that

$$\lambda g(z) = \psi(z) g(\varphi(z))$$

for all z in Δ. As above, we see that

$$\frac{d^m}{dz^m} [\psi(z) g(\varphi(z))] \bigg|_{z=w_0}$$

equals $\psi(w_0)\varphi'(w_0)^m g^{(m)}(w_0)$, plus terms involving lower order derivatives of g at w_0. Therefore

$$\lambda g^{(m)}(w_0) = \psi(w_0)\varphi'(w_0)^m g^{(m)}(w_0),$$

so that $\lambda = \psi(w_0)\varphi'(w_0)^m$. □

Propositions 4.1 and 4.2, taken together, allow us precisely to determine the spectrum of a compact operator $R_{\varphi,\psi}$.

Proposition 4.3 *Suppose that* $\mathcal{M}$ *is an infinite set. If* φ *is not an automorphism of* $\mathbb{D}$ *and the operator* $R_{\varphi,\psi}$ *is compact, then the spectrum of the operator is precisely* $\{0\} \cup \{\psi(w_0)\varphi'(w_0)^m\}_{m\in\mathcal{M}}$.

Proof. Since the functions x_m are linearly independent for different values of m, the Banach space $\mathcal{X}$ is necessarily infinite dimensional; hence the compact operator $R_{\varphi,\psi}$ is not invertible, meaning that 0 belongs to its spectrum. Every nonzero element of the spectrum is an eigenvalue and hence must equal $\psi(w_0)\varphi'(w_0)^m$, where m denotes the zero of the corresponding eigenfunction; that is, it must belong to the set $\{\psi(w_0)\varphi'(w_0)^m\}_{m\in\mathcal{M}}$. On the other hand, if $\psi(w_0)$ is nonzero, then Proposition 4.1 shows that $\{\psi(w_0)\varphi'(w_0)^m\}_{m\in\mathcal{M}}$ belongs to the spectrum of $R_{\varphi,\psi}$; if $\psi(w_0) = 0$, then every value $\psi(w_0)\varphi'(w_0)^m$ is also 0, which we already know to be in the spectrum. □

4.2 An application

Consider the weighted Bergman space A^2_α for some $\alpha \geq -1$ (where A^2_{-1}, as always, denotes the Hardy space H^2). We now apply the results of Section 4.1 to the operator $C^*_\varphi C_\varphi : A^2_\alpha \to A^2_\alpha$, namely to the case where φ has the form $\varphi(z) = az + b$. Suppose, first of all, that both a and b are nonzero and that $|a| + |b| < 1$. Proposition 2.6 guarantees that C_φ, and hence $C^*_\varphi C_\varphi$, is a compact operator on A^2_α; Observation 3.4 dictates that

$$\left(C^*_\varphi C_\varphi(f)\right)(z) = \left(\frac{1}{1-\bar{b}z}\right)^{\alpha+2} f(\tau(z)),$$

where $\sigma(z) = \bar{a}z/(-\bar{b}z + 1)$ and

$$\tau(z) = \varphi(\sigma(z)) = \frac{\left(|a|^2 - |b|^2\right) z + b}{-\bar{b}z + 1}.$$

In other words, $C_\varphi^* C_\varphi$ is a compact weighted composition operator. By direct calculation, we see that the Denjoy–Wolff point of τ is

$$w_0 = \frac{1 - |a|^2 + |b|^2 - \sqrt{\left(1 - |a|^2 + |b|^2\right)^2 - 4\,|b|^2}}{2\bar{b}}.$$

(See Section 5.1 for a brief discussion of the Denjoy–Wolff point.) Since $\|\tau\|_\infty \leq \|\varphi\|_\infty < 1$, the point w_0 must lie inside $\mathbb{D}$. Hence $C_\varphi^* C_\varphi : A_\alpha^2 \to A_\alpha^2$ satisfies the hypotheses of Proposition 4.3. Since A_α^2 contains the polynomials, $\mathcal{M}$ is equal to the set of non-negative integers; thus the spectrum of $C_\varphi^* C_\varphi : A_\alpha^2 \to A_\alpha^2$ consists of 0, together with all values of the form

$$\begin{aligned}
\left(\frac{1}{1 - \bar{b}w_0}\right)^{\alpha+2} \tau'(w_0)^m &= \left(\frac{1}{1 - \bar{b}w_0}\right)^{\alpha+2} \left(\frac{|a|}{1 - \bar{b}w_0}\right)^{2m} = \frac{|a|^{2m}}{\left(1 - \bar{b}w_0\right)^{\alpha+2+2m}} \\
= \ |a|^{2m} &\left(1 - \frac{1 - |a|^2 + |b|^2 - \sqrt{\left(1 - |a|^2 + |b|^2\right)^2 - 4\,|b|^2}}{2}\right)^{-(\alpha+2+2m)} \\
= \ |a|^{2m} &\left(\frac{2}{1 + |a|^2 - |b|^2 + \sqrt{\left(1 - |a|^2 + |b|^2\right)^2 - 4\,|b|^2}}\right)^{\alpha+2+2m}
\end{aligned}$$

for any non-negative integer m.

Having made this observation, we can easily prove the next result:

Proposition 4.4 *Take $\alpha \geq -1$. Suppose that $\varphi : \mathbb{D} \to \mathbb{D}$ has the form $\varphi(z) = az + b$, where $|a| + |b| \leq 1$; then the operator $C_\varphi : A_\alpha^2 \to A_\alpha^2$ has norm*

$$\left(\frac{2}{1 + |a|^2 - |b|^2 + \sqrt{\left(1 - |a|^2 + |b|^2\right)^2 - 4\,|b|^2}}\right)^{(\alpha+2)/2}.$$

Proof. If either a or b equals 0, then our assertion is easy to prove; assume therefore that both terms are nonzero. In view of the preceding comments, the compact case follows immediately: when $|a|+|b|<1$, the largest element of the spectrum of $C_\varphi^* C_\varphi$ is

$$\left(\frac{2}{1+|a|^2-|b|^2+\sqrt{\left(1-|a|^2+|b|^2\right)^2-4\,|b|^2}}\right)^{\alpha+2},$$

which equals $\|C_\varphi\|^2$. Hence we need only consider the situation where $|a|+|b|=1$. In this case, our claim simply reduces to

$$\left\|C_\varphi : A_\alpha^2 \to A_\alpha^2\right\| = |a|^{-(\alpha+2)/2}.$$

For every $0<r<1$, we define the map

$$\varphi_r(z) = r\varphi(z) = (ra)z + rb.$$

Since $|ra|+|rb| = r(|a|+|b|) < 1$, our norm formula is valid for each operator C_{φ_r}. For any polynomial f, the functions $f\circ\varphi_r$ converge to $f\circ\varphi$ pointwise on the closed unit disk. Hence Fatou's lemma dictates that

$$\|C_\varphi(f)\|_{2,\alpha} \le \liminf_{r\uparrow 1} \|C_{\varphi_r}(f)\|_{2,\alpha},$$

from which we see that

$$\begin{aligned}
\|C_\varphi\| &\le \liminf_{r\uparrow 1} \|C_{\varphi_r}\| \\
&= \lim_{r\uparrow 1}\left(\frac{2}{1+|ra|^2-|rb|^2+\sqrt{\left(1-|ra|^2+|rb|^2\right)^2-4\,|rb|^2}}\right)^{(\alpha+2)/2} \\
&= |a|^{-(\alpha+2)/2}.
\end{aligned}$$

We use the same technique as Cowen [9] to obtain the reverse inequality. For

$0 < r < 1$, consider the point $w_r = r|a|b/(a|b|)$ in $\mathbb{D}$; observe that

$$\begin{aligned}
\|C_\varphi\| &\geq \limsup_{r\uparrow 1} \frac{\left\|C_\varphi^*\left(K_{w_r}\right)\right\|_{2,\alpha}}{\|K_{w_r}\|_{2,\alpha}} = \limsup_{r\uparrow 1} \frac{\left\|K_{\varphi(w_r)}\right\|_{2,\alpha}}{\|K_{w_r}\|_{2,\alpha}} \\
&= \limsup_{r\uparrow 1} \left(\frac{1-|w_r|^2}{1-|\varphi(w_r)|^2}\right)^{(\alpha+2)/2} = \lim_{r\uparrow 1}\left(\frac{1-r^2}{1-(r|a|+|b|)^2}\right)^{(\alpha+2)/2} \\
&= |a|^{-(\alpha+2)/2}.
\end{aligned}$$

Therefore our formula is valid for every map $\varphi(z) = az + b$ with $|a| + |b| \leq 1$. $\square$

As we mentioned in Chapter 2, this result was originally established by Cowen [9] for composition operators on H^2; his proof was later extended to the weighted Bergman spaces by Hurst [15]. While there are obvious differences between our proof and Cowen's, it is impossible not to notice certain similarities as well. Both arguments, for example, require the calculation of the Denjoy–Wolff point of $\tau = \varphi \circ \sigma$. Approaching Cowen's work with the correct mindset, one can interpret a significant portion of his proof as being a statement about the spectrum of $C_\varphi^* C_\varphi$.

Chapter 5

Norms of Composition Operators with Linear Fractional Symbol

Let $\varphi : \mathbb{D} \to \mathbb{D}$ be a nonconstant linear fractional map, as in (3.1):

$$\varphi(z) = \frac{az+b}{cz+d}.$$

Let σ be defined as in (3.2):

$$\sigma(z) = \frac{\overline{a}z - \overline{c}}{-\overline{b}z + \overline{d}}.$$

Our goal now is to find a set of conditions, formulated in terms of the map $\tau = \varphi \circ \sigma$, under which we can determine $\|C_\varphi\|$. We introduce these conditions in Section 5.1, establishing a representation for the norm of a corresponding composition operator (Theorem 5.11). Our technique also allows us to determine the elements on which such an operator attains its norm, a question we discuss further in Section 5.2. Section 5.3 provides several specific examples of linear fractional φ that satisfy these conditions. We conclude the chapter with a few related results, including the statement of a theorem recently obtained by the author in collaboration with three other mathematicians.

The principal results of this section also appear, in somewhat abbreviated form, in a current paper of the author [14].

5.1 The spectrum of $C_\varphi^* C_\varphi$

For a non-negative integer j, let τ_j denote the jth iterate of $\tau = \varphi \circ \sigma$; that is, τ_0 is the identity map on $\mathbb{D}$ and $\tau_{j+1} = \tau \circ \tau_j$. Throughout the next two sections, we make the following assumption:

$$\textit{There is some integer } n \geq 0 \textit{ such that } \tau_n(\varphi(0)) = 0. \tag{5.1}$$

In effect, this condition is a generalization of the case where $\varphi(0) = 0$. To avoid a triviality, we also assume that φ does not have the form $\varphi(z) = az$.

We begin with several basic observations about φ:

Lemma 5.1 *Suppose that* $\tau_n(\varphi(0)) = 0$ *and that* $\varphi(z) \neq az$*; then the map* φ *cannot be an automorphism of* $\mathbb{D}$*.*

Proof. Suppose that φ is an automorphism; in other words, φ has the form

$$\varphi(z) = s\left(\frac{w - z}{1 - \overline{w}z}\right) = \frac{sz - sw}{\overline{w}z - 1}, \tag{5.2}$$

where w is a point in $\mathbb{D}$ and s is a constant of modulus 1. The corresponding map σ may be written

$$\sigma(z) = \frac{\overline{s}z - w}{\overline{s}\overline{w}z - 1}.$$

By direct calculation, we see that

$$\tau(z) = \varphi(\sigma(z)) = \frac{|s|^2\left(1 - |w|^2\right)z}{\left(1 - |w|^2\right)} = z.$$

Therefore, since $\tau_n(\varphi(0)) = 0$, the point $\varphi(0)$ must equal 0. Considering line (5.2), we see that φ must have the form

$$\varphi(z) = -sz,$$

which is excluded by our initial assumptions. □

Since φ is not an automorphism, neither is the map $\tau = \varphi \circ \sigma$. Hence τ has a *Denjoy–Wolff point*, that is, a unique fixed point w_0 in the closed unit disk such that the iterates τ_j converge to w_0 uniformly on compact subsets of $\mathbb{D}$. (See Section 2.3 of [11] for a detailed discussion of this topic.) This fact allows us to prove the next three lemmas:

Lemma 5.2 *Suppose that* $\tau_n(\varphi(0)) = 0$ *and that* $\varphi(z) \neq az$*; then the map* φ *cannot have the form* $\varphi(z) = az + b$*.*

Proof. Suppose that $\varphi(z) = az + b$; then

$$\sigma(z) = \frac{\overline{a}z}{-\overline{b}z + 1}.$$

Observe that $\sigma(0) = 0$, which means that

$$0 = \tau_n(\varphi(0)) = \tau_n(\varphi(\sigma(0))) = \tau_{n+1}(0).$$

Consequently $\tau_{k(n+1)}(0) = 0$ for any positive integer k. Since the iterates $\tau_j(0)$ converge to w_0, we conclude that $w_0 = 0$. Hence $\tau(0) = 0$, so

$$0 = \tau(0) = \varphi(\sigma(0)) = \varphi(0),$$

which implies that φ has the form $\varphi(z) = az$. □

Lemma 5.3 *Suppose that* $\tau_n(\varphi(0)) = 0$ *and that* $\varphi(z) \neq az$*; then the point* $\tau_j(\varphi(0))$ *never equals* $\sigma^{-1}(0)$.

Proof. Suppose that $\tau_m(\varphi(0)) = \sigma^{-1}(0)$ for some integer $m \geq 0$. In this case,

$$\tau_{m+1}(\varphi(0)) = \tau(\tau_m(\varphi(0))) = \tau(\sigma^{-1}(0)) = \varphi(0).$$

Hence $\tau_{k(m+1)}(\varphi(0)) = \varphi(0)$ for any positive integer k, which implies that $w_0 = \varphi(0)$. Consequently $\tau(\varphi(0)) = \varphi(0)$, from which we see that

$$\varphi(0) = \tau_n(\varphi(0)) = 0.$$

Since τ fixes $\varphi(0) = 0$, we see that $\tau(0) = 0$ as well; it follows that $\sigma(0) = 0$. In other words, the map

$$\varphi(z) = \frac{az + b}{cz + d}$$

has both $b = 0$ and $c = 0$, which means that it has the forbidden form $\varphi(z) = (a/d)z$. □

Lemma 5.4 *Suppose that* $\tau_n(\varphi(0)) = 0$ *and that* $\varphi(z) \neq az$*; then the point* $\tau_j(\varphi(0))$ *does not equal* $\tau_m(\varphi(0))$ *if* $j \neq m$.

Proof. Suppose that $\tau_j(\varphi(0)) = \tau_m(\varphi(0))$ for $j < m$. In this case,

$$\tau_{m-j}(\varphi(0)) = \varphi(0),$$

from which it follows that

$$\tau(\tau_{m-j-1}(\varphi(0))) = \varphi(0),$$

or

$$\tau_{m-j-1}(\varphi(0)) = \sigma^{-1}(0).$$

Lemma 5.3 tells us that this situation is impossible. □

In particular, Lemma 5.4 shows that $\tau_j(\varphi(0)) \neq 0$ for $j \neq n$; thus there is no ambiguity associated with the possibility that multiple integers satisfy property (5.1). While various indices appear in different contexts during the course of this argument, we always reserve n for the unique integer such that $\tau_n(\varphi(0)) = 0$.

Let W denote the set of points $\{\tau_j(\varphi(0))\}_{j=0}^{n}$. Lemma 5.3 guarantees that equation (3.6), our usual representation for $C_\varphi^* C_\varphi$, is valid at each point in W. Consider $\mathcal{K}_W^\perp$, the subspace of H^2 consisting of all functions that vanish on W. The following observation is crucial to the results of this section:

Proposition 5.5 *The subspace $\mathcal{K}_W^\perp$ is invariant under the operator $C_\varphi^* C_\varphi$.*

Proof. Suppose that f belongs to $\mathcal{K}_W^\perp$; it follows from equation (3.6) that

$$\begin{aligned} \left(C_\varphi^* C_\varphi(f)\right)(\tau_j(\varphi(0))) &= \psi(\tau_j(\varphi(0)))f(\tau_{j+1}(\varphi(0))) + \chi(\tau_j(\varphi(0)))f(\varphi(0)) \\ &= \psi(\tau_j(\varphi(0)))f(\tau_{j+1}(\varphi(0))). \end{aligned}$$

For $0 \leq j \leq n-1$, the term $f(\tau_{j+1}(\varphi(0)))$ equals 0; for $j = n$, the term $\psi(\tau_j(\varphi(0))) = \psi(0)$ vanishes. Therefore $\left(C_\varphi^* C_\varphi\right)(f)$ also belongs to $\mathcal{K}_W^\perp$. □

It is not difficult to determine the spectrum of the operator $C_\varphi^* C_\varphi : \mathcal{K}_W^\perp \to \mathcal{K}_W^\perp$, at least whenever C_φ is compact:

Proposition 5.6 *Suppose that the operator $C_\varphi : H^2 \to H^2$ is compact; then the spectrum of $C_\varphi^* C_\varphi : \mathcal{K}_W^\perp \to \mathcal{K}_W^\perp$ is precisely $0 \cup \{\psi(w_0)\tau'(w_0)^m\}_{m=0}^{\infty}$, where w_0 denotes the Denjoy–Wolff point of $\tau = \varphi \circ \sigma$.*

Proof. Since any function in $\mathcal{K}_W^\perp$ vanishes at $\varphi(0)$, the operator $C_\varphi^* C_\varphi : \mathcal{K}_W^\perp \to \mathcal{K}_W^\perp$ may be written

$$\left(C_\varphi^* C_\varphi(f)\right)(z) = \psi(z)f(\tau(z)).$$

for all z in the open set $\mathbb{D} \setminus \{\sigma^{-1}(0)\}$. The compactness of C_φ implies that $\|\tau\|_\infty \leq \|\varphi\|_\infty < 1$; hence w_0, the Denjoy–Wolff point of τ, lies in $\mathbb{D}$. If $w_0 = \sigma^{-1}(0)$, then

$$\sigma^{-1}(0) = \tau(\sigma^{-1}(0)) = \varphi(0),$$

which contradicts Lemma 5.3; consequently w_0 belongs to $\mathbb{D} \setminus \{\sigma^{-1}(0)\}$. In other words, the operator $C_\varphi^* C_\varphi : \mathcal{K}_W^\perp \to \mathcal{K}_W^\perp$ locally resembles a weighted composition operator on the set $\mathbb{D} \setminus \{\sigma^{-1}(0)\}$, in the sense we defined in Section 4.1.

Lemma 5.4 implies that τ does not fix any point $\tau_j(\varphi(0))$, which means that w_0 does not belong to the set $W = \{\tau_j(\varphi(0))\}_{j=0}^n$. Therefore, for any integer $m \geq 0$, the function

$$x_m(z) = (z - w_0)^m \prod_{j=0}^{n} (z - \tau_j(\varphi(0)))$$

belongs to $\mathcal{K}_W^\perp$ and has a zero of order m at w_0. Consequently every non-negative integer belongs to the set $\mathcal{M}$, so our claim follows from Proposition 4.3. □

This fact, although interesting, does not particularly help us to determine $\|C_\varphi\|$. Proposition 5.6 does, however, prove to be relevant to our work in Section 6.3

We now turn our attention to $\mathcal{K}_W$, the span of the kernel functions $\left\{K_{\tau_j(\varphi(0))}\right\}_{j=0}^n$; observe that it has dimension $n + 1$. Since $C_\varphi^* C_\varphi : H^2 \to H^2$ is self-adjoint, the subspace $\mathcal{K}_W$ is also invariant under the operator. Our strategy for finding $\|C_\varphi\|$ centers around determining the spectrum, namely the eigenvalues, of the operator $C_\varphi^* C_\varphi : \mathcal{K}_W \to \mathcal{K}_W$.

The next several results pertain to the eigenvalues and eigenfunctions of $C_\varphi^* C_\varphi$. The following proposition serves as a generalization of Observation 2.4:

Proposition 5.7 *Let λ be an eigenvalue of $C_\varphi^* C_\varphi : H^2 \to H^2$ with a corresponding eigenfunction g. For every integer $j \geq 0$, the following relationship holds:*

$$\begin{aligned}\lambda^{j+1} g(0) &= \left[\prod_{m=0}^{j-1} \psi(\tau_m(\varphi(0)))\right] g(\tau_j(\varphi(0))) \\ &\quad + \sum_{k=0}^{j-1} \chi(\tau_k(\varphi(0))) \left[\prod_{m=0}^{k-1} \psi(\tau_m(\varphi(0)))\right] \lambda^{j-k} g(0),\end{aligned}$$

where we take $\prod_{m=0}^{-1}(\cdot)$ to equal 1 *and $\sum_{k=0}^{-1}(\cdot)$ to equal* 0*.*

Proof (by induction). Since $\lambda g(0) = g(\varphi(0))$, the claim holds for $j = 0$. For any $j \geq 0$, equation (3.6) dictates that

$$\begin{aligned}\lambda g(\tau_j(\varphi(0))) &= \left(\left(C_\varphi^* C_\varphi\right) g\right)(\tau_j(\varphi(0))) \\ &= \psi(\tau_j(\varphi(0))) g(\tau_{j+1}(\varphi(0))) + \chi(\tau_j(\varphi(0)))\lambda g(0). \qquad (5.3)\end{aligned}$$

Now assume that our claim holds for the index j. Multiplying both sides of the consequent equation by λ and substituting expression (5.3) for $\lambda g(\tau_j(\varphi(0)))$, we obtain

$$\begin{aligned}\lambda^{j+2} g(0) &= \left[\prod_{m=0}^{j-1} \psi(\tau_m(\varphi(0)))\right] [\psi(\tau_j(\varphi(0))) g(\tau_{j+1}(\varphi(0))) + \chi(\tau_j(\varphi(0)))\lambda g(0)] \\ &\quad + \sum_{k=0}^{j-1} \chi(\tau_k(\varphi(0))) \left[\prod_{m=0}^{k-1} \psi(\tau_m(\varphi(0)))\right] \lambda^{j+1-k} g(0) \\ &= \left[\prod_{m=0}^{j} \psi(\tau_m(\varphi(0)))\right] g(\tau_{j+1}(\varphi(0))) \\ &\quad + \sum_{k=0}^{j} \chi(\tau_k(\varphi(0))) \left[\prod_{m=0}^{k-1} \psi(\tau_m(\varphi(0)))\right] \lambda^{j+1-k} g(0).\end{aligned}$$

Hence our claim also holds for the index $j+1$. $\square$

Since both $\mathcal{K}_W$ and $\mathcal{K}_W^\perp$ are invariant under $C_\varphi^* C_\varphi : H^2 \to H^2$, each eigenvalue λ of $C_\varphi^* C_\varphi$ has an eigenfunction belonging to one of the two subspaces. The next proposition provides a distinguishing characteristic for eigenfunctions in $\mathcal{K}_W^\perp$.

Proposition 5.8 *Let g be an eigenfunction for $C_\varphi^* C_\varphi : H^2 \to H^2$; then g belongs to $\mathcal{K}_W^\perp$ if and only if $g(0) = 0$.*

Proof. If g belongs to $\mathcal{K}_W^\perp$, then by definition $g(0) = g(\tau_n(\varphi(0)))$ equals 0. Conversely, suppose that g is an eigenfunction for $C_\varphi^* C_\varphi$ with $g(0) = 0$. In this case, Proposition 5.7 dictates that

$$0 = \lambda^{j+1} g(0) = \left[\prod_{m=0}^{j-1} \psi(\tau_m(\varphi(0))) \right] g(\tau_j(\varphi(0)))$$

for all $j \geq 0$. Since $\psi(\tau_m(\varphi(0)))$ is nonzero for $0 \leq m \leq n-1$, the function g must vanish on the entire set $\{\tau_j(\varphi(0))\}_{j=0}^n$. In other words, g belongs to the subspace $\mathcal{K}_W^\perp$. □

Corollary 5.9 *Suppose that g_1 and g_2 are eigenfunctions for $C_\varphi^* C_\varphi$ which belong to $\mathcal{K}_W$; if they correspond to the same eigenvalue, then one is a scalar multiple of the other.*

Proof. We appeal to the proof of Corollary 2.3, bearing in mind that no eigenfunction in $\mathcal{K}_W$ can vanish at 0. □

Consequently every eigenspace of $C_\varphi^* C_\varphi : \mathcal{K}_W \to \mathcal{K}_W$ has dimension 1. Since $C_\varphi^* C_\varphi : \mathcal{K}_W \to \mathcal{K}_W$ is a self-adjoint operator on a finite dimensional space, we know that $\mathcal{K}_W$ is spanned by eigenfunctions of $C_\varphi^* C_\varphi$. Since $\mathcal{K}_W$ has dimension $n+1$, the operator $C_\varphi^* C_\varphi : \mathcal{K}_W \to \mathcal{K}_W$ must have $n+1$ distinct eigenvalues.

We return to the result of Proposition 5.7. Taking $j = n$ and observing that $\chi(\tau_n(\varphi(0))) = \chi(0) = 1$, we obtain the expression

$$\lambda^{n+1} g(0) = \sum_{k=0}^{n} \chi(\tau_k(\varphi(0))) \left[\prod_{m=0}^{k-1} \psi(\tau_m(\varphi(0))) \right] \lambda^{n-k} g(0).$$

Suppose that the eigenfunction g belongs to $\mathcal{K}_W$; then $g(0) \neq 0$ and

$$\lambda^{n+1} = \sum_{k=0}^{n} \chi(\tau_k(\varphi(0))) \left[\prod_{m=0}^{k-1} \psi(\tau_m(\varphi(0))) \right] \lambda^{n-k}.$$

In other words, any eigenvalue λ of $C_\varphi^* C_\varphi : \mathcal{K}_W \to \mathcal{K}_W$ is a solution to this polynomial equation. Since there are $n+1$ distinct eigenvalues and the equation has no more than $n+1$ roots, we conclude that every solution is an eigenvalue. In other words,

$$p(\lambda) = \lambda^{n+1} - \sum_{k=0}^{n} \chi(\tau_k(\varphi(0))) \left[\prod_{m=0}^{k-1} \psi(\tau_m(\varphi(0))) \right] \lambda^{n-k} \tag{5.4}$$

is the characteristic polynomial of the operator $C_\varphi^* C_\varphi : \mathcal{K}_W \to \mathcal{K}_W$.

Finally, we make an observation regarding the essential norm of C_φ. (The author is indebted to Paul Bourdon for suggesting the proof of this proposition.)

Proposition 5.10 *Under the assumptions of this section,* $\|C_\varphi\|_e < 1$.

Proof. If $\|\varphi\|_\infty < 1$, then C_φ is compact, so our claim holds. Suppose then that $\|\varphi\|_\infty = 1$; since φ is not an automorphism, there is precisely one pair of points ζ and ω on $\partial\mathbb{D}$ with $\varphi(\zeta) = \omega$. Observation 3.2 shows that $\sigma(\omega) = \zeta$; Bourdon, Levi, Narayan, and Shapiro [5] proved in general that $\sigma'(\omega) = (\varphi'(\zeta))^{-1}$. Therefore $\tau(\omega) = \omega$ and $\tau'(\omega) = 1$. Consider the map $\tau_n \circ \varphi$. The assumptions of this section guarantee that $\tau_n \circ \varphi$ fixes the origin; since φ is not surjective, $\tau_n \circ \varphi$ cannot be an automorphism of $\mathbb{D}$. Hence Lemma 7.33 in [11] shows that

$$\liminf_{z \to \zeta} \frac{1 - |(\tau_n \circ \varphi)(z)|}{1 - |z|} > 1.$$

Since $\tau_n \circ \varphi$ is analytic on a neighborhood of the closed unit disk and $(\tau_n \circ \varphi)(\zeta) = \tau_n(\omega) = \omega$, the Julia–Carathéodory theorem (see Theorem 2.44 in [11]) dictates that $|(\tau_n \circ \varphi)'(\zeta))| > 1$. Consequently

$$1 < |(\tau_n)'(\varphi(\zeta)) \cdot \varphi'(\zeta)| = |(\tau_n)'(\omega) \cdot \varphi'(\zeta)| = |\varphi'(\zeta)|.$$

Appealing to expression (2.4), we see that

$$\|C_\varphi\|_e^2 = \max\left\{|\varphi'(w)|^{-1} : |w| = |\varphi(w)| = 1\right\} = |\varphi'(\zeta)|^{-1} < 1,$$

as we had hoped to show. □

Since $\|C_\varphi\| \geq 1$, Proposition 1.3 dictates that $C_\varphi : H^2 \to H^2$ attains its norm on an element of H^2; that is, $\|C_\varphi\|^2$ is an eigenvalue of $C_\varphi^* C_\varphi : H^2 \to H^2$. Proposition 2.2 guarantees that any corresponding eigenfunction must belong to $\mathcal{K}_W$. In other words, $\|C_\varphi\|^2$ is the largest eigenvalue of $C_\varphi^* C_\varphi : \mathcal{K}_W \to \mathcal{K}_W$, meaning that it is the largest zero of the polynomial p. Hence we have proved the following result:

Theorem 5.11 *Let $\varphi : \mathbb{D} \to \mathbb{D}$ be a linear fractional map, with $\varphi(z) \neq az$. Suppose that $\tau_n(\varphi(0)) = 0$ for some integer $n \geq 0$; then $\|C_\varphi\|^2$ is the largest zero of the polynomial p in equation (5.4), and the elements on which C_φ attains its norm are linear combinations of the kernel functions $\left\{K_{\tau_j(\varphi(0))}\right\}_{j=0}^{n}$.*

Whenever $n \geq 1$, Theorem 3.7 dictates that $S_\varphi < \|C_\varphi\|$. Assuming that we can find examples of such φ, this would appear to be the first case of a composition operator whose norm we can calculate, for which the norm is not given by the action of C_φ on the reproducing kernel functions of H^2.

5.2 The eigenfunctions of $C_\varphi^* C_\varphi$

Having determined a particular eigenvalue λ of $C_\varphi^* C_\varphi : \mathcal{K}_W \to \mathcal{K}_W$, it is possible to find the corresponding eigenfunctions. In particular, considering Theorem 5.11, we can identify the functions on which the operator C_φ attains its norm. Let λ be such an eigenvalue and g be its unique eigenfunction in $\mathcal{K}_W$ with $g(0) = g(\tau_n(\varphi(0))) = 1$. We write

$$g(z) = \sum_{i=0}^{n} \frac{\alpha_i}{1 - \overline{\tau_i(\varphi(0))}z},$$

where we hope to determine the coefficients α_i. For any index $0 \leq j \leq n-1$, we may appeal to Proposition 5.7 to find $g(\tau_j(\varphi(0)))$ explicitly in terms of λ:

$$g(\tau_j(\varphi(0))) = \frac{\lambda^{j+1} - \sum_{k=0}^{j-1} \chi(\tau_k(\varphi(0))) \left[\prod_{m=0}^{k-1} \psi(\tau_m(\varphi(0)))\right] \lambda^{j-k}}{\prod_{m=0}^{j-1} \psi(\tau_m(\varphi(0)))}.$$

Therefore we obtain the matrix equation

$$\left[\frac{1}{1 - \overline{\tau_i(\varphi(0))}\tau_j(\varphi(0))}\right]_{0\leq j,i\leq n} [\alpha_i]_{0\leq i\leq n} = [g(\tau_j(\varphi(0)))]_{0\leq j\leq n}.$$

The $(n+1)\times(n+1)$ matrix is simply the Gram matrix of the vectors $\left\{K_{\tau_i(\varphi(0))}\right\}_{i=0}^{n}$, whose determinant is positive since the kernel functions are linearly independent (see Theorem XX.1 in [19]). Hence we can use Cramer's rule to solve explicitly for the coefficients.

For example, take $n = 1$. Then

$$\begin{bmatrix} \frac{1}{1-|\varphi(0)|^2} & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \end{bmatrix} = \begin{bmatrix} g(\varphi(0)) \\ g(0) \end{bmatrix} = \begin{bmatrix} \lambda \\ 1 \end{bmatrix},$$

so

$$\alpha_0 = \frac{\begin{vmatrix} \lambda & 1 \\ 1 & 1 \end{vmatrix}}{\begin{vmatrix} \frac{1}{1-|\varphi(0)|^2} & 1 \\ 1 & 1 \end{vmatrix}} = \frac{\lambda - 1}{\frac{1}{1-|\varphi(0)|^2} - 1} \text{ and } \alpha_1 = \frac{\begin{vmatrix} \frac{1}{1-|\varphi(0)|^2} & \lambda \\ 1 & 1 \end{vmatrix}}{\begin{vmatrix} \frac{1}{1-|\varphi(0)|^2} & 1 \\ 1 & 1 \end{vmatrix}} = \frac{\frac{1}{1-|\varphi(0)|^2} - \lambda}{\frac{1}{1-|\varphi(0)|^2} - 1}.$$

5.3 Examples

It is not difficult to find examples of φ with the property that $\tau(\varphi(0)) = 0$. In terms of its coefficients, a linear fractional map $\varphi : \mathbb{D} \to \mathbb{D}$ (which does not have the form $\varphi(z) = az$) satisfies this condition if and only if

$$|d|^2 - |b|^2 = \frac{a}{b}\left(\bar{c}d - \bar{a}b\right).$$

In such a case, the polynomial p in (5.4) may be written

$$p(\lambda) = \lambda^2 - \chi(\varphi(0))\lambda - \psi(\varphi(0)).$$

Hence any eigenvalue λ of $C_\varphi^* C_\varphi : \mathcal{K}_W \to \mathcal{K}_W$ has the form

$$\lambda = \frac{\chi(\varphi(0)) \pm \sqrt{\chi(\varphi(0))^2 + 4\psi(\varphi(0))}}{2} = \frac{\frac{a\bar{c}d}{b} \pm \sqrt{\left(\frac{a\bar{c}d}{b}\right)^2 - 4\left(\overline{ad} - \overline{bc}\right)ad}}{2\left(|d|^2 - |b|^2\right)}.$$

In particular, $\|C_\varphi\|^2$ is the larger of these two values.

For example, take

$$\varphi(z) = \frac{16z + 8}{19z + 32}.$$

Since $\|\varphi\|_\infty < 1$, the operator C_φ is compact. Observe that $\tau(\varphi(0)) = 0$, which means that

$$\|C_\varphi\|^2 = \frac{19 + \sqrt{181}}{30} \approx 1.081787468.$$

Moreover, the operator attains its norm on the element

$$\left(-11 + \sqrt{181}\right) K_{1/4} + 13 - \sqrt{181}.$$

At this point, we turn our attention to a larger class of examples. Let n be a positive integer and r a real number greater than n. Define

$$\varphi(z) = \frac{rz - n}{-(n+1)z + (r+1)}.$$

In order to show that φ is a self-map of $\mathbb{D}$, we shift our attention momentarily to the right half-plane $\{z : \mathrm{Re}(z) > 0\}$. Consider the map

$$L(z) = \frac{1+z}{1-z},$$

a bijection that takes $\mathbb{D}$ onto the right half-plane. A simple calculation shows that

$$\left(L \circ \varphi \circ L^{-1}\right)(z) = \left(\frac{r-n}{r+n+1}\right) z + \frac{1}{r+n+1}.$$

Since both $(r-n)/(r+n+1)$ and $1/(r+n+1)$ are positive numbers, the map $L \circ \varphi \circ L^{-1}$ takes the right half-plane into itself. Consequently φ takes $\mathbb{D}$ into $\mathbb{D}$. Furthermore, since

$$|\varphi(1)| = \frac{r-n}{r-n} = 1$$

and

$$|\varphi(-1)| = \frac{r+n}{r+n+2} < 1,$$

we see that $\partial\varphi(\mathbb{D}) \cap \partial\mathbb{D} = \{1\}$.

The essential norm of the corresponding composition operator is easy to calculate:

$$\begin{aligned} \|C_\varphi\|_e^2 &= \max\left\{|\varphi'(w)|^{-1} : |w| = |\varphi(w)| = 1\right\} \\ &= |\varphi'(1)|^{-1} = \frac{(r-n)^2}{r(r+1) - n(n+1)} = \frac{r-n}{r+n+1}. \end{aligned}$$

The following observation will allow us to determine $\|C_\varphi\|$ as well:

Lemma 5.12 *Each iterate τ_j has the form*

$$\tau_j(z) = \frac{(r+n-j+1)\,z + j}{-jz + (r+n+j+1)}.$$

Proof (by induction). Observe that

$$\tau_0(z) = z = \frac{(r+n-0+1)\,z + 0}{-0z + (r+n+0+1)},$$

so the claim holds for $j = 0$. Assume then that the assertion holds for a particular index j. Since

$$\sigma(z) = \frac{rz + (n+1)}{nz + (r+1)},$$

the map $\tau = \varphi \circ \sigma$ may be written

$$\tau(z) = \frac{(r^2 - n^2)\, z + (r-n)}{(-r+n)z + (r^2 - n^2 + 2r - 2n)} = \frac{(r+n)z + 1}{-z + (r+n+2)}.$$

Note that

$$\tau_{j+1}(z) = (\tau \circ \tau_j)(z),$$

which we may write as

$$\frac{az + b}{cz + d}$$

where

$$\begin{aligned}
a &= (r+n)(r+n-j+1) + (1)(-j), \\
b &= (r+n)(j) + (1)(r+n+j+1), \\
c &= (-1)(r+n-j+1) + (r+n+2)(-j), \\
d &= (-1)(j) + (r+n+2)(r+n+j+1).
\end{aligned}$$

Performing routine manipulations, we see that

$$\begin{aligned}
\tau_{j+1}(z) &= \frac{(r+n+1)(r+n-j)z + (r+n+1)(j+1)}{(r+n+1)(-j-1)z + (r+n+1)(r+n+j+2)} \\
&= \frac{(r+n-j)z + (j+1)}{(-j-1)z + (r+n+j+2)}.
\end{aligned}$$

Therefore the claim also holds for the index $j + 1$. □

Lemma 5.12 dictates that

$$\tau_j(\varphi(0)) = \frac{(r+n-j+1)\left(-\frac{n}{r+1}\right) + j}{-j\left(-\frac{n}{r+1}\right) + (r+n+j+1)} = \frac{j-n}{r+j+1}.$$

In particular $\tau_n(\varphi(0)) = 0$, so Theorem 5.11 applies for any such φ. Observe that

$$\begin{aligned}\psi(\tau_j(\varphi(0))) &= \frac{\left(r\left(r+1\right)-n\left(n+1\right)\right)\left(\frac{j-n}{r+j+1}\right)}{\left(r\left(\frac{j-n}{r+j+1}\right)+n+1\right)\left(n\left(\frac{j-n}{r+j+1}\right)+r+1\right)} \\ &= \frac{\left(r-n\right)\left(j-n\right)\left(r+j+1\right)}{\left(j+1\right)\left(r+n+1\right)\left(r+j-n+1\right)}\end{aligned}$$

and

$$\chi(\tau_j(\varphi(0))) = \frac{n+1}{r\left(\frac{j-n}{r+j+1}\right)+n+1} = \frac{\left(n+1\right)\left(r+j+1\right)}{\left(j+1\right)\left(r+n+1\right)}.$$

Hence the characteristic polynomial of $C_\varphi^* C_\varphi : \mathcal{K}_W \to \mathcal{K}_W$ may be written

$$p(\lambda) = \lambda^{n+1} - \sum_{k=0}^{n} \frac{\left(n+1\right)\left(r+k+1\right)}{\left(k+1\right)\left(r+n+1\right)}\left[\prod_{m=0}^{k-1} \frac{\left(r-n\right)\left(m-n\right)\left(r+m+1\right)}{\left(m+1\right)\left(r+n+1\right)\left(r+m-n+1\right)}\right]\lambda^{n-k},$$

and $\left\|C_\varphi\right\|^2$ is the largest zero of this polynomial.

In particular, if $n = 1$ then

$$\left\|C_\varphi\right\|^2 = \frac{r+1}{r+2} + \frac{1}{r+2}\sqrt{\frac{2(r+1)}{r}}.$$

For $n = 2$, we solve the resulting cubic equation to obtain

$$\left\|C_\varphi\right\|^2 = \frac{r+1}{r+3} + \frac{2}{r+3}\sqrt[3]{\frac{3\left(r+1\right)}{r\left(r-1\right)}}\operatorname{Re}\left(\sqrt[3]{\left(r+4\right)+i\left(r-2\right)\sqrt{\frac{2(r+2)}{r-1}}}\right),$$

where we take the principal branch of the cube root function. For example, if

$$\varphi(z) = \frac{4z-2}{-3z+5}$$

then

$$\left\|C_\varphi\right\|^2 = \frac{5}{7} + \frac{2\operatorname{Re}\sqrt[3]{10+5i}}{7} = \frac{5}{7} + \frac{2\sqrt{5}}{7}\cos\left(\frac{\arctan\left(\frac{1}{2}\right)}{3}\right) \approx 1.345547525.$$

5.4 Further results

The strategy we used to prove Theorem 5.11 cannot be employed, at least without substantial modification, to determine $\|C_\varphi\|$ for any other linear fractional φ. For example, it is generally impossible to decompose H^2 into the direct sum of two nontrivial subspaces $\mathcal{K}_W$ and $\mathcal{K}_W^\perp$, each invariant under the operator $C_\varphi^* C_\varphi$. To be more precise, we have the following result:

Proposition 5.13 *Let $\varphi : \mathbb{D} \to \mathbb{D}$ be a non-automorphic linear fractional map which does not have the form $\varphi(z) = az + b$. Let W be a collection of points in $\mathbb{D}$ such that both $\mathcal{K}_W$ and $\mathcal{K}_W^\perp$ are nontrivial subspaces of H^2. Suppose that $\mathcal{K}_W$ is invariant under the operator $C_\varphi^* C_\varphi : H^2 \to H^2$; then $\tau_n(\varphi(0)) = 0$ for a unique integer $n \geq 0$, there are $n + 1$ points in W, and $W = \{\tau_j(\varphi(0))\}_{j=0}^{n}$.*

Taking W to be a singleton set, we obtain a variation of the statement of Proposition 3.6. Not surprisingly, the proofs of these two propositions involve the same basic idea.

Proof of Proposition 5.13. The orthogonal complement $\mathcal{K}_W^\perp$ is also invariant under the self-adjoint operator $C_\varphi^* C_\varphi$. We use this fact to determine the structure of W. To make our argument more comprehensible, we break the proof into several smaller claims:

Claim 5.14 *The point $\sigma^{-1}(0)$ does not belong to W.*

Proof. Suppose that $\sigma^{-1}(0) = \overline{c}/\overline{a}$ does belong to W. Line (3.7) dictates that

$$\left(C_\varphi^* C_\varphi(f)\right)(\sigma^{-1}(0)) = \frac{\overline{c}}{\overline{a}} f'(\varphi(0))\tau'(\sigma^{-1}(0)) + \frac{\overline{ad}}{\overline{ad} - \overline{bc}} f(\varphi(0)),$$

which must equal 0 for any f in $\mathcal{K}_W^\perp$. Let f_1 be a function in $\mathcal{K}_W^\perp$ which has a zero of order 1 at each point in W, a zero of order 1 at $\varphi(0)$, and is nonzero at every other point in $\mathbb{D}$. (It is possible, of course, that $\varphi(0)$ belongs to W, in which case f_1 simply has a zero of order 1 at each point in W.) The assumption that $\varphi(z) \neq az+b$ guarantees that $c \neq 0$; since $f_1(\varphi(0)) = 0$ and $\tau = \varphi\circ\sigma$ is univalent, the term $f_1'(\varphi(0))$ must equal 0, which is not the case. $\Diamond$

It follows that equation (3.6) is valid at each point y in W, which means that

$$0 = \left(C_\varphi^* C_\varphi(f)\right)(y) = \psi(y) f(\tau(y)) + \chi(y) f(\varphi(0))$$

for all f in $\mathcal{K}_W^\perp$.

Claim 5.15 *If y is a nonzero element of W, then $\tau(y)$ also belongs to W.*

Proof. Again consider the function f_1 in $\mathcal{K}_W^\perp$, as defined in the proof of Claim 5.14. Since $f_1(\varphi(0)) = 0$ and $\psi(y) \neq 0$, it follows that $f_1(\tau(y))$ must also vanish. Therefore $\tau(y)$ belongs to the set $W \cup \{\varphi(0)\}$. If $\tau(y) = \varphi(0)$, then y would equal $\sigma^{-1}(0)$, contradicting Claim 5.14. Therefore $\tau(y)$ must belong to W. $\Diamond$

Claim 5.16 *The point $\varphi(0)$ belongs to W.*

Proof. Suppose first that 0 is the only point in W. In this case, the span of the constant function $K_0(z) = 1$ is invariant under $C_\varphi^* C_\varphi$. Therefore

$$\left(C_\varphi^* C_\varphi\right)(K_0) = C_\varphi^*(K_0) = K_{\varphi(0)}$$

is a constant function; hence $\varphi(0)$ equals 0, which belongs to W. Now assume that W contains a nonzero element y; suppose that $\varphi(0)$ does not belong to W. Consider a function f_2 in $\mathcal{K}_W^\perp$ which does not vanish at any point in $\mathbb{D} \setminus W$. Claim 5.15 shows that $\tau(y)$ also belongs to W, which means that

$$0 = \left(C_\varphi^* C_\varphi(f_2)\right)(y) = \psi(y) f_2(\tau(y)) + \chi(y) f_2(\varphi(0)) = \chi(y) f_2(\varphi(0)).$$

Since $\chi(y) \neq 0$ and $f_2(\varphi(0)) \neq 0$, we obtain a contradiction. Therefore $\varphi(0)$ must belong to W. $\Diamond$

Claim 5.17 *There is a unique integer* $n \geq 0$ *such that* $\tau_n(\varphi(0)) = 0$*; moreover, every point in the set* $\{\tau_j(\varphi(0))\}_{j=0}^{n}$ *belongs to* W.

Proof. If $\varphi(0) = 0$, then this fact is apparent; thus we consider the situation where $\varphi(0) \neq 0$. Suppose that $\tau_j(\varphi(0)) \neq 0$ for all j. Since $\varphi(0)$ belongs to W, Claim 5.15 shows that every point $\tau_j(\varphi(0))$ belongs to W. The sequence $\{\tau_j(\varphi(0))\}_{j=0}^{\infty}$ converges to w_0, the Denjoy–Wolff point of τ. In order for $\mathcal{K}_W^{\perp}$ to be nontrivial, the set W must have no limit point in $\mathbb{D}$; suppose then that w_0 lies on $\partial\mathbb{D}$. In this case, Lemma 3.3 in [4] dictates that there is some positive constant C such that

$$1 - |\tau_j(\varphi(0))| \geq Cj^{-1}$$

for all $j \geq 1$. Therefore $\{\tau_j(\varphi(0))\}_{j=0}^{\infty}$ is not a Blaschke sequence, which means that there is no element of H^2, other than the zero function, which vanishes on the set (see Theorem 2.3 in [13]). Hence the subspace $\mathcal{K}_W^{\perp}$ must be trivial, which contradicts our initial assumptions.

In other words, there is some integer $n \geq 0$ such that $\tau_n(\varphi(0)) = 0$. Since φ does not have the form $\varphi(z) = az$, Lemma 5.4 guarantees that n is the unique integer with this property. Since $\tau_j(\varphi(0)) \neq 0$ for $0 \leq j \leq n-1$, Claim 5.15 shows that every point in the set $\{\tau_j(\varphi(0))\}_{j=0}^{n}$ also belongs to W. $\Diamond$

Claim 5.18 *Every point in* W *belongs to the set* $\{\tau_j(\varphi(0))\}_{j=0}^{n}$.

Proof. Let y be an element of W. Repeating the argument of Claim 5.17, we see that the points $\tau_j(y)$ cannot all be nonzero; let m be the smallest integer such that

$\tau_m(y) = 0$. Suppose that $m > n$. Observe that $\tau_{m-n}(y) = \varphi(0)$, or

$$\tau_{m-n-1}(y) = \sigma^{-1}(0).$$

If $m - n - 1 = 0$, then $y = \sigma^{-1}(0)$; if $m - n - 1 > 0$, then Claim 5.15 shows that $\tau_{m-n-1}(y) = \sigma^{-1}(0)$ belongs to W. In either case, we obtain a contradiction to Claim 5.14. Consequently $m \leq n$, which means that $y = \tau_{n-m}(\varphi(0))$. $\Diamond$

Combining these observations, we obtain the desired conclusion. $\square$

While we cannot decompose H^2 as we did in Section 5.1, it is possible to adapt the basic techniques of that argument to obtain an "$n = \infty$" version of Theorem 5.11. We sketch several results along these lines, all of which appear in a current paper of Paul Bourdon, Erin Fry, Christina Spofford, and the author [4]. The following theorem may be considered our main result:

Theorem 5.19 *Let* φ *be a linear fractional self-map of* $\mathbb{D}$ *that fixes the point* 1*. If* $\|C_\varphi\| > \|C_\varphi\|_e$*, then* $\|C_\varphi\|^2$ *is the largest value of* λ *which satisfies the equation*

$$\sum_{k=0}^{\infty} \chi(\tau_k(\varphi(0))) \left[\prod_{m=0}^{k-1} \psi(\tau_m(\varphi(0)))\right] \left(\frac{1}{\lambda}\right)^{k+1} = 1. \tag{5.5}$$

We provide a rudimentary outline of the proof, omitting several important details; the complete argument appears in [4].

Outline of the proof of Theorem 5.19. Let $\lambda > \|C_\varphi\|_e^2$ be an eigenvalue for $C_\varphi^* C_\varphi$ which has a corresponding eigenfunction g that does vanish at 0. Checking the necessary details, we obtain the same functional equation as in Section 5.1:

$$\begin{aligned} \lambda^{j+1} g(0) &= \left[\prod_{m=0}^{j-1} \psi(\tau_m(\varphi(0)))\right] g(\tau_j(\varphi(0))) \\ &\quad + \sum_{k=0}^{j-1} \chi(\tau_k(\varphi(0))) \left[\prod_{m=0}^{k-1} \psi(\tau_m(\varphi(0)))\right] \lambda^{j-k} g(0). \end{aligned}$$

Dividing both sides by λ^{j+1}, we see that

$$\begin{aligned} g(0) &= \left[\prod_{m=0}^{j-1} \psi(\tau_m(\varphi(0)))\right] \left(\frac{1}{\lambda}\right)^{j+1} g(\tau_j(\varphi(0))) \\ &\quad + \sum_{k=0}^{j-1} \chi(\tau_k(\varphi(0))) \left[\prod_{m=0}^{k-1} \psi(\tau_m(\varphi(0)))\right] \left(\frac{1}{\lambda}\right)^{k+1} g(0). \end{aligned}$$

We can show that the term

$$\left[\prod_{m=0}^{j-1} \psi(\tau_m(\varphi(0)))\right] \left(\frac{1}{\lambda}\right)^{j+1} g(\tau_j(\varphi(0)))$$

tends to 0 as j goes to infinity. Therefore, since $g(0) \neq 0$, the eigenvalue λ satisfies equation (5.5).

On the other hand, if $\lambda > \|C_\varphi\|_e^2$ is a solution to (5.5), we can prove that it must be an eigenvalue for $C_\varphi^* C_\varphi$. If λ were not an eigenvalue, then it would not belong to the spectrum of $C_\varphi^* C_\varphi$; in other words, the operator $C_\varphi^* C_\varphi - \lambda$ would be invertible. Using this fact, we can demonstrate that

$$\sum_{k=0}^{\infty} h(\tau_k(\varphi(0))) \left[\prod_{m=0}^{k-1} \psi(\tau_m(\varphi(0)))\right] \left(\frac{1}{\lambda}\right)^{k+1} = 0$$

for all h in H^2. We can show that this equation contradicts (5.5).

Since $\|C_\varphi\|_e < \|C_\varphi\|$, Proposition 1.3 dictates that $\|C_\varphi\|^2$ is the largest eigenvalue for $C_\varphi^* C_\varphi$; Proposition 2.2 guarantees that the corresponding eigenfunctions do not vanish at 0. Consequently $\|C_\varphi\|^2$ satisfies equation (5.5) and is greater than every other solution. □

In the case where $\tau_n(\varphi(0)) = 0$ for some integer $n \geq 0$, line (5.5) reduces to the polynomial equation from Theorem 5.11. Theorems 5.11 and 5.19 are similar in spirit: they both describe $\|C_\varphi\|$ in terms of the zeros of a particular analytic function.

Theorem 5.19 has several interesting consequences. For example, let

$$\varphi(z) = \frac{\alpha - 1}{\alpha - z}$$

for some $\alpha > 1$; it is not difficult to show that $\|C_\varphi\| > \|C_\varphi\|_e$. For such a φ, equation (5.5) may be rewritten

$$\sum_{k=0}^{\infty} \frac{(\alpha-1)^{k+1}}{(k+1)\alpha - 1} \left(\frac{1}{\lambda}\right)^{k+1} = 1. \tag{5.6}$$

This representation for $\|C_\varphi\|$ allows us to prove an astonishing fact. Take $\alpha = 2$, so that $\varphi(z) = 1/(2 - z)$; in this case, line (5.6) reduces to

$$\sum_{k=0}^{\infty} \frac{1}{2k+1} \left(\frac{1}{\lambda}\right)^{k+1} = 1.$$

Summing the series and taking $\lambda = \|C_\varphi\|^2$, we see that

$$\frac{1}{2\|C_\varphi\|} \log\left(\frac{\|C_\varphi\| + 1}{\|C_\varphi\| - 1}\right) = 1. \tag{5.7}$$

If $\|C_\varphi\|$ happened to be an algebraic number, then Baker's theorem [3] would dictate that the left-hand side of (5.7) must be transcendental; since 1 is algebraic, we conclude that the norm of C_φ is, in fact, a transcendental number. This example shows that there can be no algebraic formula, involving only the coefficients of φ, which gives $\|C_\varphi\|$ for every linear fractional φ. In particular, Proposition 4.4 cannot be extended in an obvious manner to include all linear fractional maps.

The second half of the proof of Theorem 5.19 yields a helpful test for determining the relationship between $\|C_\varphi\|$ and $\|C_\varphi\|_e$:

Corollary 5.20 *Let φ be a non-automorphic linear fractional self-map of $\mathbb{D}$ that fixes the point 1. If there is a number $\Lambda > \|C_\varphi\|_e^2$ with*

$$\sum_{k=0}^{\infty} \chi(\tau_k(\varphi(0))) \left[\prod_{m=0}^{k-1} \psi(\tau_m(\varphi(0)))\right] \left(\frac{1}{\Lambda}\right)^{k+1} = 1,$$

then Λ *is an eigenvalue for* $C_\varphi^* C_\varphi$*; in particular,* $\|C_\varphi\|^2 \geq \Lambda > \|C_\varphi\|_e^2$.

For example, we can apply this test to a map of the form

$$\varphi(z) = \frac{(r+s)z + (1-s)}{r(1-s)z + (1+sr)},$$

where $0 < s < 1$ and $-1 < r < 0$, to show that $\|C_\varphi\|_e$ is strictly less than $\|C_\varphi\|$. For any such φ, we know that the spectral radius of C_φ is equal to $\|C_\varphi\|_e$ (see Theorem 3.9 in [11]). Consequently C_φ^* cannot be a hyponormal operator, which answers a question posed by Cowen and MacCluer [12].

One can also prove a version of Theorem 5.19 which holds for certain compact composition operators. In this case, the criterion guaranteeing the validity of our representation for $\|C_\varphi\|$ is no longer that $\|C_\varphi\|_e < \|C_\varphi\|$ (which would always hold, since $\|C_\varphi\|_e = 0$); instead the condition is that $|\psi(w_0)| < \|C_\varphi\|^2$, where w_0 denotes the Denjoy–Wolff point of τ.

Chapter 6

Norms of Composition Operators on Subspaces of H^2

Consider the Hardy space H^2. For an integer $n \geq 0$, we write $H^2_{(n)}$ to denote the subspace of H^2 consisting of all functions with a zero of order at least n at the origin. Throughout this chapter, we take $\varphi : \mathbb{D} \to \mathbb{D}$ to be a nonconstant analytic map with $\varphi(0) = 0$. Observe that each subspace $H^2_{(n)}$ is invariant under the corresponding composition operator C_φ. While it is easy to determine the norm of C_φ on H^2 (namely $\|C_\varphi\| = 1$), there is no obvious method for finding the norm of C_φ on $H^2_{(n)}$. In this chapter, we approach this question using the techniques we have already developed. Not surprisingly, we obtain particularly interesting results in the case where φ is a linear fractional map.

6.1 The essential norm of C_φ on $H^2_{(n)}$

As it turns out, it is not difficult to determine the essential norm of C_φ on the subspace $H^2_{(n)}$. We state the following proposition without proof; this fact, which appears as Proposition 5.1 in [26], serves as an important step in the proof of Shapiro's essential norm formula:

Proposition 6.1 *Let T be a bounded operator on a Hilbert space $\mathcal{H}$. Let $\{K_m\}$ be a sequence of compact self-adjoint operators on $\mathcal{H}$; write $P_m = I - K_m$, where I denotes the identity operator on $\mathcal{H}$. Suppose that $\|P_m\| = 1$ for each m and that $\lim_{m\to\infty} \|P_m(h)\| = 0$ for each h in $\mathcal{H}$; then $\|T\|_e = \lim_{m\to\infty} \|TP_m\|$.*

In particular, take $\mathcal{H}$ to be $H^2_{(n)}$. For $m \geq n+1$, let K_m denote the orthogonal projection of $H^2_{(n)}$ onto the subspace spanned by the monomials $\left\{z^k\right\}_{k=n}^{m-1}$. Each K_m, having finite dimensional range, is compact. Observe that $P_m = I - K_m$ is the projection of $H^2_{(n)}$ onto $H^2_{(m)}$; therefore $\|P_m\| = 1$ for every m. For any function $f(z) = \sum_{k=n}^{\infty} a_k z^k$ in $H^2_{(n)}$, we see that

$$\|P_m(f)\|_2^2 = \sum_{k=m}^{\infty} |a_k|^2$$

for all $m \geq n+1$; hence $\lim_{m\to\infty} \|P_m(f)\|_2 = 0$. In other words, the operators K_m and P_m satisfy the hypotheses of Proposition 6.1.

Since $(C_\varphi P_m)(f) = C_\varphi(f)$ for every f in $H^2_{(m)}$ and $\|P_m\| = 1$, we note that

$$\left\|C_\varphi P_m : H^2_{(n)} \to H^2_{(n)}\right\| = \left\|C_\varphi : H^2_{(m)} \to H^2_{(m)}\right\|$$

for any $m \geq n+1$. Considering Proposition 6.1, we arrive at the following conclusion:

Observation 6.2 *Let φ be an analytic self-map of $\mathbb{D}$ with $\varphi(0) = 0$; then*

$$\left\|C_\varphi : H^2_{(n)} \to H^2_{(n)}\right\|_e = \lim_{m\to\infty} \left\|C_\varphi : H^2_{(m)} \to H^2_{(m)}\right\|.$$

Since this observation holds for any value of n, the next result follows immediately:

Observation 6.3 *Let φ be an analytic self-map of $\mathbb{D}$ with $\varphi(0) = 0$. For any integer $n \geq 0$, the essential norm of the operator $C_\varphi : H^2_{(n)} \to H^2_{(n)}$ is equal to the essential norm of $C_\varphi : H^2 \to H^2$.*

For the rest of this chapter, without risk of ambiguity, we write $\|C_\varphi\|_e$ to denote the essential norm of the operator C_φ acting on any of the subspaces $H^2_{(n)}$.

6.2 The norm of C_φ on $H^2_{(n)}$

At this point, we commence our investigation of the norm of $C_\varphi : H^2_{(n)} \to H^2_{(n)}$. We begin by introducing more manageable notation. Suppose that $H^2_{(n)}$ is invariant under some operator $T : H^2 \to H^2$; we write $\|T\|_{(n)}$ to denote the quantity

$$\left\|T : H^2_{(n)} \to H^2_{(n)}\right\|.$$

Given a map φ, we are interested in describing the behavior of $\|C_\varphi\|_{(n)}$ as n varies. If $m \leq n$, then $H^2_{(n)}$ is contained in $H^2_{(m)}$, so $\|C_\varphi\|_{(n)}$ is certainly less than or equal to $\|C_\varphi\|_{(m)}$. In many cases, we can say a good deal more. For example, suppose that φ is an inner function; then (by Proposition 2.1) the operator C_φ is an isometry of H^2, which means that $\|C_\varphi\|_{(n)} = 1$ for every $n \geq 0$. Applying Observation 6.2, we obtain the following result:

Observation 6.4 *Suppose that $\varphi : \mathbb{D} \to \mathbb{D}$ is an inner function with $\varphi(0) = 0$; then $\|C_\varphi\|_{(n)} = \|C_\varphi\|_e = 1$ for every $n \geq 0$.*

This situation, however, is somewhat atypical. When $m < n$, it is quite often the case that $\|C_\varphi\|_{(n)} < \|C_\varphi\|_{(m)}$. The next proposition allows us to discuss this issue more precisely:

Proposition 6.5 *Let $\varphi : \mathbb{D} \to \mathbb{D}$ be an analytic map with $\varphi(0) = 0$. Suppose, for some $n \geq 1$, that $\|C_\varphi\|_{(n)}$ is strictly greater than $\|C_\varphi\|_e$; then*

$$\|C_\varphi\|_{(n)} < \|C_\varphi\|_{(n-1)}\,.$$

Proof. Since $C_\varphi : H^2_{(n)} \to H^2_{(n)}$ is not extremally noncompact, Proposition 1.3 dictates that the operator attains its norm on some unit vector g in $H^2_{(n)}$. Note that the function $\widetilde{g}(z) = g(z)/z$ is a unit vector in the subspace $H^2_{(n-1)}$. In light of our hypotheses, Observation 6.4 shows that φ cannot be an inner function; hence

$$\lim_{r \uparrow 1} \left| g\big(\varphi\big(re^{i\theta}\big)\big) \right| < \lim_{r \uparrow 1} \left| \frac{g\big(\varphi\big(re^{i\theta}\big)\big)}{\varphi\big(re^{i\theta}\big)} \right|$$

for θ in a set of positive measure. Therefore

$$\|C_\varphi\|_{(n)} = \|C_\varphi(g)\|_2 \; < \|C_\varphi(\widetilde{g})\|_2 \leq \|C_\varphi\|_{(n-1)}\,,$$

as we had hoped to show. □

Appealing to this proposition, we obtain a new proof of a related result, originally due to Joel Shapiro [28]:

Proposition 6.6 *Suppose that $\varphi : \mathbb{D} \to \mathbb{D}$ is analytic with $\varphi(0) = 0$; then φ is inner if and only if $\|C_\varphi\|_{(1)} = 1$.*

Proof. Observation 6.4 shows that $\|C_\varphi\|_{(1)} = 1$ whenever φ is inner. Suppose then that φ is not inner; Proposition 4.1 in [28] dictates that

$$\|C_\varphi\|_e < \sqrt{\frac{1 + |\varphi(0)|}{1 - |\varphi(0)|}} = 1.$$

Since the norm of an operator is never less than its essential norm, $\|C_\varphi\|_{(1)} \geq \|C_\varphi\|_e$. If $\|C_\varphi\|_{(1)} = \|C_\varphi\|_e$, then $\|C_\varphi\|_{(1)} < 1$. If $\|C_\varphi\|_{(1)} > \|C_\varphi\|_e$, then Proposition 6.5 shows that $\|C_\varphi\|_{(1)} < \|C_\varphi\|_{(0)} = 1$. □

In other words, whenever φ is not inner, the norm of C_φ acting on $H^2_{(1)}$ is strictly less than its norm on $H^2_{(0)} = H^2$. As there is no obvious characteristic distinguishing this pair of subspaces from any other, one might expect to find an analogous result for other values of n; for example, perhaps $\|C_\varphi\|_{(1)} > \|C_\varphi\|_{(2)}$ whenever φ is not inner. We shall see, however, that this phenomenon does not actually occur; Section 6.3 will provide specific counterexamples.

In light of Observation 6.2, we see that the values $\|C_\varphi\|_{(n)}$ form a non-increasing sequence converging to $\|C_\varphi\|_e$. Therefore Proposition 6.5 shows that one of the following two situations must hold:

(1) The sequence $\|C_\varphi\|_{(n)}$ is strictly decreasing, with $\|C_\varphi\|_{(n)} > \|C_\varphi\|_e$ for each n and $\lim_{n\to\infty} \|C_\varphi\|_{(n)} = \|C_\varphi\|_e$.

(2) The sequence $\|C_\varphi\|_{(n)}$ is strictly decreasing up to some integer $N \geq 0$, with $\|C_\varphi\|_{(n)} = \|C_\varphi\|_{(N)} = \|C_\varphi\|_e$ for all $n \geq N$.

As shown by Observation 6.4, any inner function φ provides a rather trivial example of situation (2). If φ is not inner, then either situation can manifest itself. For example, if the operator $C_\varphi : H^2 \to H^2$ is compact, then situation (1) must hold, since $\|C_\varphi\|_{(n)}$ never equals 0. In the next section, we shall encounter non-inner φ which serve as further examples of situation (2).

6.3 The linear fractional case

As our final topic, we consider a composition operator C_φ whose symbol

$$\varphi(z) = \frac{az}{cz+1} \tag{6.1}$$

is a nonconstant linear fractional self-map of $\mathbb{D}$ with $\varphi(0) = 0$; without loss of generality, we have taken the coefficient d to equal 1. As in Section 3.1, we define the map

$$\sigma(z) = \overline{a}z - \overline{c}.$$

Recall that φ is a self-map of $\mathbb{D}$ if and only if σ is as well; that is, if and only if $|a| + |c| \leq 1$. The operator C_φ is noncompact if and only if $\|\varphi\|_\infty = 1$; that is, if and only if there are points ζ and ω on $\partial\mathbb{D}$ such that $\varphi(\zeta) = \omega$. Observation 3.2 shows that this situation occurs if and only if there are ζ and ω on $\partial\mathbb{D}$ such that $\sigma(\omega) = \zeta$, which occurs precisely when $|a| + |c| = 1$.

Whenever C_φ is noncompact, it is easy to determine its essential norm:

Lemma 6.7 *Let $\varphi : \mathbb{D} \to \mathbb{D}$ be a nonconstant linear fractional self-map of $\mathbb{D}$ with $\varphi(0) = 0$, as in (6.1). Suppose that $|a| + |c| = 1$, so that the operator C_φ is noncompact; then $\|C_\varphi\|_e^2 = |a|$.*

Proof. If $|a| = 1$ and $c = 0$, then φ is inner, so $\|C_\varphi\|_e^2 = 1 = |a|$. If c is nonzero, then there is exactly one point ζ on $\partial\mathbb{D}$ such that

$$1 = |\varphi(\zeta)| = \left|\frac{a\zeta}{c\zeta + 1}\right|.$$

In this case, expression (2.4) shows that

$$\begin{aligned}
\|C_\varphi\|_e^2 &= \max\left\{|\varphi'(w)|^{-1} : |w| = |\varphi(w)| = 1\right\} \\
&= |\varphi'(\zeta)|^{-1} = \frac{|c\zeta + 1|^2}{|a|} = \left|\frac{c\zeta + 1}{a\zeta}\right|^2 \left|a\zeta^2\right| = |a|,
\end{aligned}$$

which proves our assertion. □

We are interested in determining $\|C_\varphi\|_{(n)}$ for various values of n, in particular $n = 1$. As usual, our strategy involves considering the spectrum of the operator

$C_\varphi^* C_\varphi$, which we write in terms of the functions

$$\begin{aligned}\tau(z) &= \varphi(\sigma(z)),\\ \psi(z) &= \frac{\overline{a}z}{\overline{a}z-\overline{c}},\\ \chi(z) &= \frac{\overline{c}}{-\overline{a}z+\overline{c}}.\end{aligned}$$

We need to be somewhat careful, though, about the precise definition of C_φ^*. Even though each subspace $H^2_{(n)}$ is invariant under C_φ, the same is not necessarily true for C_φ^*. The adjoint of C_φ on $H^2_{(n)}$ is, in fact, the restriction of the operator $P_n C_\varphi^*$ to $H^2_{(n)}$, where C_φ^* denotes the adjoint of C_φ on H^2 and P_n is the orthogonal projection of H^2 onto $H^2_{(n)}$; we write $C_\varphi^{(*,n)}$ to denote this operator.

For one particular set of maps, $C_\varphi^{(*,n)}$ actually does equal the restriction of C_φ^* to $H^2_{(n)}$, a fact which allows us to determine $\|C_\varphi\|_{(n)}$:

Proposition 6.8 *If φ has the form $\varphi(z) = az$, then $\|C_\varphi\|_{(n)} = |a|^n$ for any n.*

Proof. If $|a| = 1$, then C_φ is an inner function, which guarantees that $\|C_\varphi\|_{(n)} = 1$ for any n. Now consider the case where $0 < |a| < 1$, for which the operator C_φ is compact. Cowen's adjoint formula (Proposition 3.3) dictates C_φ^*, the adjoint of C_φ on H^2, may be written C_σ, where $\sigma(z) = \overline{a}z$. Note that each subspace $H^2_{(n)}$ is invariant under C_φ^*, which means that $C_\varphi^{(*,n)}$ is simply the restriction of $C_\varphi^* = C_\sigma$. Hence the operator $C_\varphi^{(*,n)} C_\varphi : H^2_{(n)} \to H^2_{(n)}$ may be written C_τ, where $\tau(z) = |a|^2 z$.

Applying the results of Chapter 4, we can determine the spectrum of $C_\varphi^{(*,n)} C_\varphi = C_\tau$ precisely. Observe that 0 is the unique fixed point of the map $\tau : \mathbb{D} \to \mathbb{D}$. For any $m \geq n$, the subspace $H^2_{(n)}$ contains the monomial z^m, a function with a zero of order m at the origin; by definition, $H^2_{(n)}$ does not contain any function with a zero of order less than n at 0. Hence the set $\mathcal{M}$ consists exactly of the integers which are greater than or equal to n. Proposition 4.3 shows that the spectrum of C_τ on $H^2_{(n)}$ is precisely $0 \cup \left\{|a|^{2m}\right\}_{m=n}^{\infty}$, which means that $\|C_\varphi\|^2_{(n)} = \|C_\tau\|_{(n)} = |a|^{2n}$. $\square$

For any f in H^2, observe that

$$\left(C_\varphi^*(f)\right)(0) = \left\langle C_\varphi^*(f), K_0\right\rangle = \left\langle f, C_\varphi(K_0)\right\rangle = \left\langle f, K_0\right\rangle = f(0).$$

Consequently the subspace $H^2_{(1)}$ is invariant under C_φ^*, so $C_\varphi^{(*,1)}$ is actually just the restriction of C_φ^* to $H^2_{(1)}$. This fact, which is specific to this particular subspace, is crucial to our next result:

Proposition 6.9 *Let $\varphi : \mathbb{D} \to \mathbb{D}$ be a nonconstant linear fractional map with $\varphi(0) = 0$, as in (6.1), which does not have the form $\varphi(z) = az$. If the operator $C_\varphi : H^2 \to H^2$ is compact, then $\|C_\varphi\|^2_{(1)} = \psi(w_0)$, where w_0 denotes the Denjoy–Wolff point of $\tau = \varphi \circ \sigma$.*

Proof. Since $\tau_0(\varphi(0)) = \varphi(0) = 0$, the map φ is an example of the class discussed in Section 5.1. In this instance, the subspace $\mathcal{K}_W^\perp = \mathcal{K}_{\{0\}}^\perp$ is identical to $H^2_{(1)}$. Therefore Proposition 5.6 shows that the spectrum of $C_\varphi^* C_\varphi : \mathcal{K}_W^\perp \to \mathcal{K}_W^\perp$, in other words $C_\varphi^{(*,1)} C_\varphi : H^2_{(1)} \to H^2_{(1)}$, is precisely $\{0\} \cup \{\psi(w_0)\tau'(w_0)^m\}_{m=0}^{\infty}$. The largest element of the spectrum is $\psi(w_0)$, which equals $\|C_\varphi\|^2_{(1)}$. $\square$

Proposition 6.9 allows us to determine $\|C_\varphi\|_{(1)}$ for any linear fractional φ.

Theorem 6.10 *Let*

$$\varphi(z) = \frac{az}{cz+1}$$

be a nonconstant linear fractional self-map of $\mathbb{D}$ with $\varphi(0) = 0$; then

$$\|C_\varphi\|^2_{(1)} = \frac{2\,|a|^2}{1 + |a|^2 - |c|^2 + \sqrt{\left(1 - |a|^2 + |c|^2\right)^2 - 4\,|c|^2}}.$$

Proof. If the coefficient c equals 0, then φ has the form $\varphi(z) = az$; in this case, our claim reduces to saying that $\|C_\varphi\|_{(1)}^2 = |a|^2$, which follows from Proposition 6.8. For the rest of the proof, we assume that $c \neq 0$.

Suppose first of all that C_φ is compact; that is, $|a| + |c| < 1$. Proposition 6.9 tells us that $\|C_\varphi\|_{(1)}^2 = \psi(w_0)$, where w_0 denotes the Denjoy–Wolff point of

$$\tau(z) = \frac{|a|^2 z - a\overline{c}}{(\overline{a}c)z + \left(1 - |c|^2\right)}.$$

By direct calculation, we see that

$$w_0 = \frac{-1 + |a|^2 + |c|^2 + \sqrt{\left(1 - |a|^2 - |c|^2\right)^2 - 4\,|ac|^2}}{2\overline{a}c}.$$

Therefore

$$\begin{aligned} \|C_\varphi\|_{(1)}^2 &= \psi(w_0) = \frac{\overline{a}w_0}{\overline{a}w_0 - \overline{c}} = \frac{-2\overline{a}cw_0}{-2\overline{a}cw_0 + 2\,|c|^2} \\ &= \frac{1 - |a|^2 - |c|^2 - \sqrt{\left(1 - |a|^2 - |c|^2\right)^2 - 4\,|ac|^2}}{1 - |a|^2 + |c|^2 - \sqrt{\left(1 - |a|^2 - |c|^2\right)^2 - 4\,|ac|^2}}. \end{aligned} \tag{6.2}$$

Performing elementary manipulations, we see that line (6.2) equals

$$\begin{aligned} &\frac{2\,|a|^2}{1 + |a|^2 - |c|^2 + \sqrt{\left(1 - |a|^2 - |c|^2\right)^2 - 4\,|ac|^2}} \\ =\; &\frac{2\,|a|^2}{1 + |a|^2 - |c|^2 + \sqrt{\left(1 - |a|^2 + |c|^2\right)^2 - 4\,|c|^2}}, \end{aligned}$$

thus proving our result for compact C_φ.

We now consider a map that induces a noncompact composition operator; that is, φ with $|a| + |c| = 1$. In this case, our claim simply asserts that $\|C_\varphi\|_{(1)}^2 = |a|$. For every $0 < r < 1$, we define the map

$$\varphi_r(z) = r\varphi(z) = \frac{(ra)z}{cz + 1}.$$

Since $|ra| + |c| < |a| + |c| = 1$, our norm formula is valid for each operator C_{φ_r}. For any polynomial f, the functions $f \circ \varphi_r$ converge to $f \circ \varphi$ pointwise on the closed unit disk. Hence Fatou's lemma dictates that

$$\|C_\varphi(f)\|_2 \leq \liminf_{r \uparrow 1} \|C_{\varphi_r}(f)\|_2 \, ,$$

from which we see that

$$\begin{aligned} \|C_\varphi\|_{(1)}^2 &\leq \liminf_{r \uparrow 1} \|C_{\varphi_r}\|_{(1)}^2 \\ &= \lim_{r \uparrow 1} \frac{2\,|ra|^2}{1 + |ra|^2 - |c|^2 + \sqrt{\left(1 - |ra|^2 + |c|^2\right)^2 - 4\,|c|^2}} \\ &= |a|. \end{aligned}$$

We appeal to Lemma 6.7 to establish the reverse inequality:

$$\|C_\varphi\|_{(1)}^2 \geq \|C_\varphi\|_e^2 = |a|.$$

Therefore our formula is valid for every linear fractional $\varphi : \mathbb{D} \to \mathbb{D}$ with $\varphi(0) = 0$. $\square$

Combining the results of Lemma 6.7 and Theorem 6.10, we make the following observation:

Observation 6.11 *Let $\varphi : \mathbb{D} \to \mathbb{D}$ be a nonconstant linear fractional self-map of $\mathbb{D}$ with $\varphi(0) = 0$, as in (6.1). Suppose that $|a| + |c| = 1$, so that the operator C_φ is noncompact; then $\|C_\varphi\|_{(1)}^2 = \|C_\varphi\|_e^2 = |a|$.*

In light of this fact, our next result follows immediately:

Theorem 6.12 *Let $\varphi : \mathbb{D} \to \mathbb{D}$ be a nonconstant linear fractional self-map of $\mathbb{D}$ with $\varphi(0) = 0$, as in (6.1). Suppose that $|a| + |c| = 1$, so that the operator C_φ is noncompact; then $\|C_\varphi\|_{(n)}^2 = \|C_\varphi\|_e^2 = |a|$ for all $n \geq 1$.*

Proof. We know that $\|C_\varphi\|_e \leq \|C_\varphi\|_{(n)} \leq \|C_\varphi\|_{(1)}$ for any integer $n \geq 1$. Since $\|C_\varphi\|_{(1)} = \|C_\varphi\|_e$, all of these quantities must be equal. □

In other words, these maps provide a nontrivial example of situation (2), as described at the end of Section 6.2. They also serve as a counterexample to the hypothesis that $\|C_\varphi\|_{(1)} > \|C_\varphi\|_{(2)}$ for any non-inner φ.

The astute reader will notice a definite resemblance between Theorem 6.10 and Proposition 4.4; the methods of proof and the actual norm formulae are strikingly similar. This correspondence is not accidental. An alternate proof of Theorem 6.10 helps to shed light on this situation:

Alternate proof of Theorem 6.10. In his proof of Theorem 5 in [9], Cowen showed that the adjoint $C_\varphi^{(*,1)} : H^2_{(1)} \to H^2_{(1)}$ is unitarily equivalent to the operator $\overline{a}C_\sigma : H^2 \to H^2$, where $\sigma(z) = \overline{a}z - \overline{c}$. Hence Proposition 4.4 dictates that

$$\begin{aligned} \|C_\varphi\|^2_{(1)} &= \left\|C_\varphi^{(*,1)}\right\|^2_{(1)} = |a|^2 \|C_\sigma\|^2_{(0)} \\ &= \frac{2|a|^2}{1 + |a|^2 - |c|^2 + \sqrt{\left(1 - |a|^2 + |c|^2\right)^2 - 4|c|^2}} \end{aligned}$$

for any such φ. □

This point of view also provides further insight into Observation 6.11. Reworded slightly, that result simply states that the operator $C_\varphi : H^2_{(1)} \to H^2_{(1)}$ is extremally noncompact. This fact makes more sense once we recognize that $\overline{a}C_\sigma : H^2 \to H^2$, which is unitarily equivalent to $C_\varphi^{(*,1)} : H^2_{(1)} \to H^2_{(1)}$, possesses the same property.

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List of Symbols

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k_w	4
$\mathcal{K}_W$	4
$\mathcal{K}_W^\perp$	4
C_φ	4
$\|\cdot\|_e$	7
$r_e(\cdot)$	7
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S_φ^*	9
S_φ	9
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